XXIV International Seminar
Nonlinear Phenomena in
Complex Systems

General classical and quantum description of electromagnetic interactions of spinning particles in Riemannian spacetimes

Yuri N. Obukhov*, Alexander J. Silenko+¤, Oleg V. Teryaev¤ɷ

*Nuclear Safety Institute, RAS, Moscow, Russia

+Research Institute for Nuclear Problems, BSU, Minsk, Belarus

¤Joint Institute for Nuclear Research, Dubna, Russia

ɷNational Research Nuclear University ``MEPhI'', Moscow, Russia

May 16-19, 2017

Joint Institute for Power and Nuclear Research – Sosny

OUTLINE

- Covariant Dirac equation and its general transformation to the Foldy-Wouthuysen representation for a relativistic particle
- Dirac particle in strong stationary and nonstationary gravitational fields
  - Spin in a gravitational field of a Kerr source
  - Relativistic Lense-Thirring effect
  - Riemann-Cartan spacetimes. Bounds on the Cartan torsion
- Comparison of spin behavior in electromagnetic and gravitational fields
- General description of a Dirac particle in strong gravitational and electromagnetic fields
  - Dirac particle in a gravitational wave with allowance for electromagnetic interactions
- Summary

Covariant Dirac equation and its general transformation to the Foldy-Wouthuysen representation for a relativistic particle

The quantum theory is based on the covariant Dirac equation:

$$\left(i\hbar\gamma^{\alpha}D_{\alpha}-mc+\frac{\mu'}{2c}\sigma^{\alpha\beta}F_{\alpha\beta}+\frac{\delta'}{2c}\sigma^{\alpha\beta}G_{\alpha\beta}\right)\Psi=0,\quad D_{\alpha}=e_{\alpha}^{i}D_{i},$$

$$D_{i}=\partial_{i}-\frac{ie}{\hbar}A_{i}+\frac{i}{4}\sigma^{\alpha\beta}\Gamma_{i\alpha\beta},\quad \sigma^{\alpha\beta}=\frac{i}{2}(\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha}),\quad \Gamma_{i\alpha\beta}=-\Gamma_{i\beta\alpha},$$

$$i,\alpha,\beta=0,1,2,3.$$

$$\Gamma_{i\alpha\beta}=-\Gamma_{i\beta\alpha}=e_{i}^{\gamma}\Gamma_{\gamma\alpha\beta},\qquad \Gamma_{\gamma\alpha\beta}=\frac{1}{2}\left(-C_{\gamma\alpha\beta}+C_{\alpha\beta\gamma}+C_{\beta\gamma\alpha}\right),$$

$$C_{\alpha\beta\gamma}=-C_{\beta\alpha\gamma}=e_{\alpha}^{i}e_{\beta}^{j}\left(e_{\gamma j,i}-e_{\gamma i,j}\right).$$

Isotropic stationary metric :

$$ds^{2}=V^{2}c^{2}\left(dx^{0}\right)^{2}-W^{2}\delta_{ab}\left(dx^{a}-K^{a}dx^{0}\right)\left(dx^{b}-K^{b}dx^{0}\right)$$

Schwinger gauge:

$$e_{i}^{0}=V\delta_{\mu}^{0},\quad e_{i}^{a}=W\left(\delta_{i}^{a}-K^{a}\delta_{\mu}^{0}\right)$$

Tetrad indexes are blue or are marked by hats

Nonunitary transformation of the initial equation brings the Hermitian Dirac Hamiltonian (isotropic stationary metric)

$$\mathcal{H}_D = \beta mc^2 V + \frac{c}{2}\{F, \boldsymbol{\alpha}\cdot\mathbf{p}\} + \frac{c}{2}\left(\mathbf{K}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{K}\right) + \frac{\hbar c}{4}\left(\nabla\times\mathbf{K}\right)\cdot\boldsymbol{\Sigma},$$

$$F = V/W.$$

Yu. N. Obukhov, A.J. Silenko and O.V. Teryaev, Phys. Rev. D **80**, 064044 (2009).

General metric:

$$ds^2 = V^2\left(dx^0\right)^2 - \delta_{\hat{a}\hat{b}} W_c^{\hat{a}} W_d^{\hat{b}}\left(dx^c - K^c dx^0\right)\left(dx^d - K^d dx^0\right).$$

General Hermitian Dirac Hamiltonian:

$$\mathcal{H}_D = \beta mc^2 V + e\Phi + \frac{c}{2}\alpha^a\{F_a^b, \pi_b\} + \frac{c}{2}\left(\mathbf{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\mathbf{K}\right)$$

$$+\frac{\hbar c}{4}\left(\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} - \Upsilon\gamma_5\right) - \beta V\left(\boldsymbol{\Sigma}\cdot\boldsymbol{\mathcal{M}} + i\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{P}}\right), \qquad F_a^b = VW_{\hat{a}}^b,$$

$W_{\hat{a}}^b$ is the matrix inverse to $W_b^{\hat{a}}$.

$$\Gamma_{i\,\widehat{a}\widehat{0}} = \frac{c^2}{V} W^b{}_{\widehat{a}}\, \partial_b V\, e_i{}^{\widehat{0}} - \frac{c}{V}\, \mathcal{Q}_{(\widehat{a}\widehat{b})}\, e_i{}^{\widehat{b}},$$

$$\Gamma_{i\,\widehat{a}\widehat{b}} = \frac{c}{V}\, \mathcal{Q}_{[\widehat{a}\widehat{b}]}\, e_i{}^{\widehat{0}} + \left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right) e_i{}^{\widehat{c}}.$$

$$\mathcal{Q}_{\widehat{a}\widehat{b}} = g_{\widehat{a}\widehat{c}} W^d{}_{\widehat{b}} \left(\frac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e \partial_e W^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e \partial_d K^e\right),$$

$$\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = W^d{}_{\widehat{a}} W^e{}_{\widehat{b}}\, \partial_{[d} W^{\widehat{c}}{}_{e]}, \qquad \mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = g_{\widehat{c}\widehat{d}} \mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{d}}.$$

$$\mathcal{F}^b{}_a = V W^b{}_{\widehat{a}}, \qquad \boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}$$

$$\Upsilon = V \epsilon^{\widehat{a}\widehat{b}\widehat{c}} \Gamma_{\widehat{a}\widehat{b}\widehat{c}} = -V \epsilon^{\widehat{a}\widehat{b}\widehat{c}} \mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}},$$

$$\Xi^a = \frac{V}{c} \epsilon^{\widehat{a}\widehat{b}\widehat{c}} \Gamma_{\widehat{0}\widehat{b}\widehat{c}} = \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\, \mathcal{Q}^{\widehat{b}\widehat{c}}.$$

$$\boldsymbol{\mathcal{M}}^a = \mu' \mathfrak{B}^a + \delta' \mathfrak{E}^a, \qquad \boldsymbol{\mathcal{P}}_a = c\delta' \mathfrak{B}_a - \mu' \mathfrak{E}_a / c.$$

We should carefully distinguish the *anholonomic* components $\mathfrak{E}, \mathfrak{B}$ of the Maxwell tensor, $\mathfrak{E}_a = \{F_{\widehat{1}0}, F_{\widehat{2}0}, F_{\widehat{3}0}\}$ and $\mathfrak{B}^a = \{F_{\widehat{2}\widehat{3}}, F_{\widehat{3}\widehat{1}}, F_{\widehat{1}\widehat{2}}\}$, and the *holonomic* components $\boldsymbol{E}, \boldsymbol{B}$ of F_{ij}, which are $\boldsymbol{E}_a = \{F_{10}, F_{20}, F_{30}\}$ and $\boldsymbol{B}^a = \{F_{23}, F_{31}, F_{12}\}$. For the general metric (2.3) with the tetrad (2.2), these fields are related via (denoting $w := \det W^{\widehat{c}}{}_d$)

$$\mathfrak{E}_a = \frac{1}{V} W^b{}_{\widehat{a}} (\boldsymbol{E} + c\boldsymbol{K} \times \boldsymbol{B})_b,$$

$$\mathfrak{B}^a = \frac{1}{w} W^{\widehat{a}}{}_b \boldsymbol{B}^b,$$

General form of the Dirac Hamiltonian:

$$\mathcal{H}_D = \beta\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[\beta, \mathcal{M}] = [\beta, \mathcal{E}] = 0, \quad \{\beta, \mathcal{O}\} = 0.$$

Relativistic Foldy-Wouthuysen transformation is performed by the unitary operator:

$$U = \frac{1 + \sqrt{1 + X^2} + \beta X}{\sqrt{2\sqrt{1 + X^2}(1 + \sqrt{1 + X^2})}}, \quad X = \left\{ \frac{1}{2\mathcal{M}}, \mathcal{O} \right\}.$$

The final Foldy-Wouthuysen Hamiltonian is given by

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E} + \frac{1}{4}\left\{ \frac{1}{2\epsilon^2 + \{\epsilon, \mathcal{M}\}}, (\beta[\mathcal{O},[\mathcal{O},\mathcal{M}]] - [\mathcal{O},[\mathcal{O},\mathcal{F}]]) \right\}.$$

$$\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2} \qquad \mathcal{F} = \mathcal{E} - i\hbar\frac{\partial}{\partial t}$$

A.J. Silenko, Phys. Rev. A **91**, 022103 (2015).

Dirac particle in strong stationary and nonstationary gravitational fields

Static metric

$$ds^2 = V^2 c^2 dt^2 - W^2 \delta_{ab} dx^a dx^b$$

Schwarzschild field
Accelerated frame

Foldy-Wouthuysen Hamiltonian:

$$\mathcal{H}_{FW}^{(1)} = \beta\varepsilon' - \frac{\beta\hbar mc^4}{4}\left\{\frac{1}{2\varepsilon'^2 + mc^2\{\varepsilon', V\}}, [\boldsymbol{\Sigma}\cdot(\boldsymbol{\Phi}\times\mathbf{p}) - \boldsymbol{\Sigma}\cdot(\mathbf{p}\times\boldsymbol{\Phi}) + \hbar\nabla\cdot\boldsymbol{\Phi}]\right\}$$

$$+\beta\frac{\hbar c^2}{16}\left\{\frac{1}{\varepsilon'}, [\boldsymbol{\Sigma}\cdot(\mathbf{G}\times\mathbf{p}) - \boldsymbol{\Sigma}\cdot(\mathbf{p}\times\mathbf{G}) + \hbar\nabla\cdot\mathbf{G}]\right\}$$

$$\varepsilon' = \sqrt{m^2c^4V^2 + \frac{1}{2}c^2\left\{F^2, \mathbf{p}^2\right\}}, \quad \boldsymbol{\Phi} = F^2\nabla V, \quad \mathbf{G} = \nabla\left(F^2\right)$$

Only $\hbar$ is a small parameter. All terms are exact. All terms of order of $\hbar$ and leading terms of order of $\hbar^2$ describing contact interaction are included

Any direct spin-gravity coupling proportional to $\boldsymbol{\Sigma}\cdot\boldsymbol{\Phi}$, $\boldsymbol{\Sigma}\cdot\mathbf{G}$ and violating *CP* invariance does not appear!

Stationary metric

$$ds^2 = V^2 c^2 dt^2 - W^2 \delta_{ab}\left(dx^a - K^a c dt\right)\left(dx^b - K^b c dt\right)$$

Foldy-Wouthuysen Hamiltonian:

$$\mathcal{H}_{FW} = \mathcal{H}_{FW}^{(1)} + \mathcal{H}_{FW}^{(2)},$$

$$\mathcal{H}_{FW}^{(2)} = \frac{c}{2}\left(\mathbf{K}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{K}\right) + \frac{\hbar c}{4}\boldsymbol{\Sigma}\cdot\left(\nabla\times\mathbf{K}\right)$$

$$-\frac{\hbar c^3}{16}\left\{\frac{1}{2\varepsilon'^2 + mc^2\{\varepsilon', V\}}, \left\{F^2, \boldsymbol{\Sigma}\cdot\mathbf{Q}\right\}\right\}$$

$$\mathbf{Q} = \mathbf{p}\times\nabla\left(\mathbf{K}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{K}\right) - \nabla\left(\mathbf{K}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{K}\right)\times\mathbf{p}$$

$$-\mathbf{p}\times\left(\mathbf{p}\times\left(\nabla\times\mathbf{K}\right)\right) - \left(\left(\nabla\times\mathbf{K}\right)\times\mathbf{p}\right)\times\mathbf{p}$$

Yu. N. Obukhov, A.J. Silenko and O.V. Teryaev, Phys. Rev. D **80**, 064044 (2009).

Operator equation of spin motion

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[H_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}^{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}^{(2)} \times \mathbf{\Pi}$$

Term depending on the static part of the metric

$$\mathbf{\Omega}^{(1)} = -\frac{mc^4}{2}\left\{\frac{1}{2\varepsilon'^2 + mc^2\{\varepsilon', V\}}, (\mathbf{\Phi} \times \mathbf{p} - \mathbf{p} \times \mathbf{\Phi})\right\} + \frac{c^2}{8}\left\{\frac{1}{\varepsilon'}, (\mathbf{G} \times \mathbf{p} - \mathbf{p} \times \mathbf{G})\right\}$$

$$\mathbf{\Omega}^{(2)} = \frac{c}{2}(\nabla \times \mathbf{K}) - \frac{c^3}{8}\left\{\frac{1}{2\varepsilon'^2 + mc^2\{\varepsilon', V\}}, \{F^2, \mathbf{Q}\}\right\}$$

Semiclassical equation of spin motion

$$\frac{d\mathbf{S}}{dt} = \mathbf{\Omega} \times \mathbf{S}$$

$$\mathbf{\Omega} = -\frac{mc^4}{\varepsilon'^2 + mc^2\varepsilon' V}\mathbf{\Phi} \times \mathbf{p} + \frac{c^2}{2\varepsilon'}\mathbf{G} \times \mathbf{p} + \frac{c}{2}(\nabla \times \mathbf{K}) - \frac{cF^2\mathbf{Q}}{4\left(\varepsilon'^2 + mc^2\varepsilon' V\right)}$$

This is the general solution of the problem

Classical theory with the Schwinger gauge

$$\mathbf{\Omega} = \frac{1}{FV(\gamma+1)}\mathbf{v}\times\mathbf{\Phi} - \frac{1}{2F}\mathbf{v}\times\mathbf{G} + \frac{c}{2}(\nabla\times\mathbf{K})$$

$$-\frac{\gamma}{2c(\gamma+1)}\left[\mathbf{v}\times\nabla(\mathbf{v}\cdot\mathbf{K}) + \mathbf{v}\times(\mathbf{v}\cdot\nabla)\mathbf{K}\right].$$

Since $\varepsilon' = mc^2V\gamma, \qquad \mathbf{p} = mW\gamma\mathbf{v},$

the classical and quantum mechanical equations of spin motion agree

The classical Hamiltonian $H_{class} = \varepsilon' + c\mathbf{K}\cdot\mathbf{p} + \mathbf{s}\cdot\mathbf{\Omega}$ **agrees with the quantum one**

The classical and quantum approaches are in the best agreement

Spin in a gravitational field of a Kerr source

Operator equation of spin motion

$$\mathbf{\Omega}^{(2)} = \frac{c}{2}\left(\nabla \times \mathbf{K}\right) - \frac{c^3}{8}\left\{\frac{1}{2\varepsilon'^2 + mc^2\{\varepsilon', V\}}, \left\{F^2, \mathbf{Q}\right\}\right\}$$

Lense-Thirring term

relativistic correction

Semiclassical equation of spin motion

$$\mathbf{\Omega} = -\frac{mc^4}{\varepsilon'^2 + mc^2\varepsilon' V}\mathbf{\Phi} \times \mathbf{p} + \frac{c^2}{2\varepsilon'}\mathbf{G} \times \mathbf{p} + \frac{c}{2}\left(\nabla \times \mathbf{K}\right) - \frac{cF^2\mathbf{Q}}{4\left(\varepsilon'^2 + mc^2\varepsilon' V\right)}$$

This is the relativistic Lense-Thirring effect for the spin

Foldy-Wouthuysen Hamiltonian in the most general case

$$\mathcal{H}_{FW} = \mathcal{H}_{FW}^{(1)} + \mathcal{H}_{FW}^{(2)},$$

$$\mathcal{H}_{FW}^{(1)} = \beta\epsilon' + \frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'}, \left(2e^{cae}\Pi_e\{p_b, \mathcal{F}_c^d\mathcal{F}_{a,d}^b\} - \Pi^a\{p_b, \mathcal{F}_a^b\Upsilon\}\right)\right\} +$$

$$\frac{\hbar mc^4}{4}e^{cae}\Pi_e\left\{\frac{1}{\mathcal{T}}, \{p_d, \mathcal{F}_c^d\mathcal{F}_a^b\nabla_b V\right\},$$

Yu. N. Obukhov, A.J. Silenko and O.V. Teryaev, Phys. Rev. D **84**, 024025 (2011); **88**, 084014 (2013).

$$\mathcal{H}_{FW}^{(2)} = \frac{c}{2}(\boldsymbol{K}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{K}) + \frac{\hbar c}{4}\boldsymbol{\Sigma}\cdot\boldsymbol{\Xi} +$$

$$\frac{\hbar c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left[-ce^{eaf}\Sigma_f\left\{\{p_d, \mathcal{F}_e^d\}, \left(\{p_c, \mathcal{F}_a^b\nabla_b K^c\} - \{p_b, \mathcal{F}_{a,c}^b K^c\}\right)\right\} -\right.\right.$$

$$\left.\left.\frac{c}{2}\Sigma_a\left\{\{p_d, \mathcal{F}_c^d\}, \{p_b, (\mathcal{F}^{bc}\Xi^a - \mathcal{F}^{ab}\Xi^c)\}\right\} + e^{cae}\Sigma_e\left\{\{p_d, \mathcal{F}_c^d\}, \{p_b, \dot{\mathcal{F}}_a^b\}\right\}\right]\right\},$$

$$\epsilon' = \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{ac}\{p_b, \mathcal{F}_a^b\}\{p_d, \mathcal{F}_c^d\}}, \quad \mathcal{T} = 2\epsilon'^2 + \{\epsilon', mc^2V\}.$$

Angular velocity of spin rotation:

$$\Omega^a_{(1)} = \frac{mc^4}{2}\left\{\frac{1}{\mathcal{T}}, \{p_e, \epsilon^{abc}\mathcal{F}^e{}_b\mathcal{F}^d{}_c\partial_d V\}\right\}$$
$$+\frac{c^2}{8}\left\{\frac{1}{\epsilon'}, \{p_e, (2\epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c + \delta^{ab}\mathcal{F}^e{}_b\,\Upsilon)\}\right\},$$

$$\Omega^a_{(2)} = \frac{\hbar c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left\{\{p_e, \mathcal{F}^e{}_b\}, \left\{p_f, \left[\epsilon^{abc}\left(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\} + \frac{c}{2}\Xi^a.$$

The classical limit is ($\mathcal{E}$ and $\mathcal{B}$ are the Pomeransky-Khriplovich gravitoelectromagnetic fields)

$$\boldsymbol{\Omega} = -\boldsymbol{\mathcal{B}} + \frac{\gamma}{\gamma+1}\frac{\boldsymbol{v}\times\boldsymbol{\mathcal{E}}}{c^2}.$$

The observed relativistic Lense-Thirring precession:

A quasi-periodic modulation of the iron line centroid energy in the black hole binary H 1743-322

Adam Ingram,[1★] Michiel van der Klis,[1] Matthew Middleton,[2] Chris Done,[3] Diego Altamirano,[4] Lucy Heil,[1] Phil Uttley[1] & Magnus Axelsson[5]

[1] *Anton Pannekoek Institute for Astronomy, University of Amsterdam, Science Park 904, 1098 XH, Amsterdam, The Netherlands.*
[2] *Institute of Astronomy, Cambridge University, Madingley Road, CB3 0HA, Cambridge, UK*
[3] *Center for Extragalactic Astronomy, Department of Physics, University of Durham, South Road, Durham, DH1 3LE, UK*
[4] *Department of Physics & Astronomy, University of Southampton, Southampton, Hampshire SO17 1BJ, UK*
[5] *Department of Physics, Tokyo Metropolitan University, Minami Osawa 1-1, Hachioji, Tokyo 192-0397, Japan*

ABSTRACT

Accreting stellar-mass black holes often show a 'Type-C' quasi-periodic oscillation (QPO) in their X-ray flux, and an iron emission line in their X-ray spectrum. The iron line is generated through continuum photons reflecting off the accretion disk, and its shape is distorted by relativistic motion of the orbiting plasma and the gravitational pull of the black hole. The physical origin of the QPO has long been debated, but is often attributed to Lense-Thirring precession, a General Relativistic effect causing the inner flow to precess as the spinning black hole twists up the surrounding space-time. This predicts a characteristic rocking of the iron line between red and blue shift as the receding and approaching sides of the disk are respectively illuminated. Here we

To make a theoretical description of such an effect, one needs only to obtain a metric of a moving Kerr source and to substitute it to our general equations of the spin motion presented below.

The gyrogravitomagnetic ratio is exactly the same for any particles: $g_{grav}=2$
This is the manifestation of the post-Einsteinian equivalence principle (equivalence principle for spin; no difference between the classical gravity and the classical limit of quantum mechanics)

Riemann-Cartan spacetimes

Cartan spacetime torsion tensor **Christoffel symbols**

$$S^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}.$$

The Cartan spacetime torsion does not influence a particle motion but affects the spin

The classical limit of the quantum-mechanical Hamiltonian obtained by Y. N. Obukhov, A. J. Silenko, and O. V. Teryaev, Phys. Rev. D **90**, 124068 (2014) **for a Dirac particle in a magnetic field is given by**

$$H = -g_N \frac{\mu_N}{\hbar} \mathbf{B} \cdot \mathbf{s} - \boldsymbol{\omega} \cdot \mathbf{s} - \frac{c}{2} \breve{\mathbf{T}} \cdot \mathbf{s}, \qquad \breve{T}^{\alpha} = -\frac{1}{2} \eta^{\alpha\mu\nu\lambda} T_{\mu\nu\lambda},$$

where g_N is the nuclear g factor and μ_N is the nuclear magneton. $\eta^{\alpha\mu\nu\lambda}$ is the totally antisymmetric Levi-Civita tensor.

$\boldsymbol{\omega}$ is the angular velocity of the Earth rotation.

The strongest restriction extracted from the experimental data presented by C. Gemmel et al., Eur. Phys. J. D **57**, 303 (2010) **is**

$$|\check{\boldsymbol{T}}| \cdot |\cos\Theta| < 2.4 \times 10^{-15}\ \mathrm{m}^{-1}.$$

Θ is the angle between $\boldsymbol{B}$ and the torsion $\check{\boldsymbol{T}}$

About 1% of the angular velocity of the Earth rotation ($|\boldsymbol{\omega}| = 7.29 \times 10^{-5}$ rad/s).

Eur. Phys. J. D **57**, 303–320 (2010)
DOI: 10.1140/epjd/e2010-00044-5

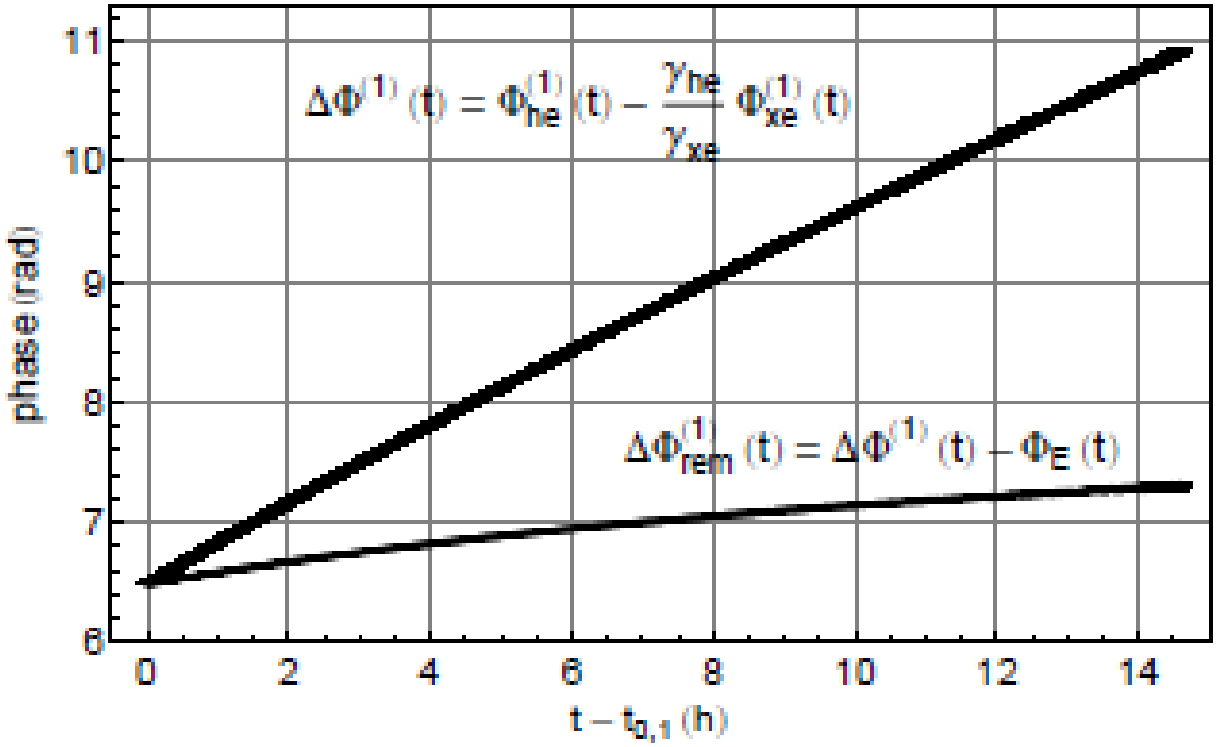
THE EUROPEAN PHYSICAL JOURNAL D

Regular Article

Ultra-sensitive magnetometry based on free precession of nuclear spins

C. Gemmel[1], W. Heil[1,a], S. Karpuk[1], K. Lenz[1], Ch. Ludwig[1], Yu. Sobolev[1,b], K. Tullney[1], M. Burghoff[2], W. Kilian[2], S. Knappe-Grüneberg[2], W. Müller[2], A. Schnabel[2], F. Seifert[2], L. Trahms[2], and St. Baeßler[3,c]

[1] Institut für Physik, 55099 Mainz, Germany
[2] Physikalisch-Technische Bundesanstalt, 10587 Berlin, Germany
[3] University of Virginia, Charlottesville, VA 22904, USA



before correction

after correction

Figure 4.14: Weighted phase difference for run (1) before ($\Delta\Phi^{(1)}$) and after ($\Delta\Phi^{(1)}_{rem}$) the subtraction of the Earth's rotation term Φ_E (Eq. (4.21)). One data point comprises 20 sub-data sets.

We have extracted the new restriction of about 0.9% from the experimental data presented in C. Gemmel et al, Eur. Phys. J. D **57**, 303 (2010).

$$\left|\chi(\mathrm{He}) - G\chi(\mathrm{Xe})\right| < 0.009, \qquad G = \frac{g_{\mathrm{He}}}{g_{\mathrm{Xe}}}$$

Comparison of spin behavior in electromagnetic and gravitational fields

$$(i\gamma^{\mu}D_{\mu}-m)\psi=0, \qquad \mu=0,1,2,3$$

The squared Dirac equation is

$$(-\hbar^2 g^{\mu\nu}D_{\mu}D_{\nu}-\hbar^2\gamma^{[\mu}\gamma^{\nu]}D_{\mu}D_{\nu}-m^2c^2)\psi=0.$$

Explicit computation yields

$$\left[\pi^{\mu}\pi_{\mu}-\underline{\frac{\hbar}{2}\sigma^{ab}\left(\frac{q}{c}F_{ab}+m\Phi_{ab}\right)}+\frac{\hbar^2}{4}R\right.$$

$$\left.+\frac{\hbar^2}{16}\left(2\Gamma^{\mu}_{ab}\Gamma^{ab}_{\mu}+ie^{abcd}\Gamma^{\mu}_{ab}\Gamma_{\mu cd}\gamma_5\right)-m^2c^2\right]\psi=0,$$

where $F_{ab}=e^{\mu}_{a}e^{\nu}_{b}F_{\mu\nu}$ are the tensorlike electromagnetic field coefficients and $\Phi_{ab}=\frac{1}{2m}\left\{\pi^{\mu},\Gamma_{\mu ab}\right\}$.

Φ_{ab} coincides with the spin (and momentum) transport matrix in a gravitational field.

The equation shows that the spin interactions with electromagnetic and gravitational fields are pretty similar

General description of a Dirac particle in strong gravitational and electromagnetic fields

Our approach has allowed us to perform a quantum mechanical description of a relativistic Dirac particle in arbitrary gravitational and electromagnetic fields

We find for the FW Hamiltonian:

$$\mathcal{H}_{FW} = \mathcal{H}_{FW}^{(1)} + \mathcal{H}_{FW}^{(2)} + \mathcal{H}_{FW}^{(3)} + \mathcal{H}_{FW}^{(4)}.$$

$$\mathcal{H}_{FW}^{(1)} = \beta\epsilon' + \frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'}, \left(2\epsilon^{cae}\Pi_e\{\pi_b, \mathcal{F}^d{}_c\partial_d\mathcal{F}^b{}_a\} + \Pi^a\{\pi_b, \mathcal{F}^b{}_a\Upsilon\}\right)\right\}$$
$$+ \frac{\hbar m c^4}{4}\epsilon^{cae}\Pi_e\left\{\frac{1}{\mathcal{T}}, \{\pi_d, \mathcal{F}^d{}_c\mathcal{F}^b{}_a\partial_b V\}\right\},$$

$$\mathcal{H}_{FW}^{(2)} = \frac{c}{2}\left(K^a\pi_a + \pi_a K^a\right) + \frac{\hbar c}{4}\Sigma_a\Xi^a$$
$$+ \frac{\hbar c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\Sigma_a\{\pi_e, \mathcal{F}^e{}_b\}, \left\{\pi_f, \left[\epsilon^{abc}\left(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\},$$

$$\mathcal{H}^{(3)}_{FW} = e\Phi - \frac{e\hbar c^2}{4}\left\{\frac{1}{\epsilon'}, V^2\Pi^a\mathfrak{B}_a\right\}$$

$$+\frac{e\hbar c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left[\Sigma_a\epsilon^{abc}\left(\{\mathcal{F}^d{}_b, \pi_d\}V^2\mathfrak{E}_c - V^2\mathfrak{E}_b\{\mathcal{F}^d{}_c, \pi_d\}\right) - 2\hbar\mathcal{F}^b{}_a\partial_b(V^2\mathfrak{E}^a)\right]\right\},$$

$$\mathcal{H}^{(4)}_{FW} = -\frac{c}{8}\left\{\frac{1}{\epsilon'}, \left[\Sigma_a\epsilon^{abc}\left(\{\mathcal{F}^d{}_b, \pi_d\}V\mathcal{P}_c - V\mathcal{P}_b\{\mathcal{F}^d{}_c, \pi_d\}\right) - 2\hbar\mathcal{F}^b{}_a\partial_b(V\mathcal{P}^a)\right]\right\} - V\,\Pi^a\mathcal{M}_a$$

$$+\frac{c^2}{4}\left\{\frac{1}{\mathcal{T}}, \left(\Pi^a\left\{\{\mathcal{F}^c{}_a\mathcal{F}^d{}_b V\mathcal{M}^b, \pi_c\}, \pi_d\right\} + \beta\hbar\left\{\mathcal{F}^b{}_a[\mathcal{J}^a + K^c\partial_c(V\mathcal{P}^a)], \pi_b\right\}\right)\right\}$$

$$\epsilon' = \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{ac}\{\pi_b, \mathcal{F}^b{}_a\}\{\pi_d, \mathcal{F}^d{}_c\}}, \qquad \mathcal{T} = 2\epsilon'^2 + \{\epsilon', mc^2V\},$$

$$\mathcal{J}^a = e^{abc}\mathcal{F}^d{}_b\partial_d(V\mathcal{M}_c) + \frac{\partial\mathcal{P}^a}{c\partial t}.$$

The equation of the spin motion:

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}_{(2)} \times \mathbf{\Pi}.$$

$$
\begin{aligned}
\Omega^a_{(1)} = \frac{mc^4}{2}\left\{\frac{1}{\mathcal{T}}, \{\pi_e, \epsilon^{abc}\mathcal{F}^e{}_b\mathcal{F}^d{}_c\partial_d V\}\right\} \\
+ \frac{c^2}{8}\left\{\frac{1}{\epsilon'}, \{\pi_e, (2\epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c + \delta^{ab}\mathcal{F}^e{}_b\,\Upsilon)\}\right\} \\
- \frac{ec^2}{2}\left\{\frac{1}{\epsilon'}, V^2\mathfrak{B}_a\right\} - \frac{2V}{\hbar}\mathcal{M}_a \\
+ \frac{c^2}{2\hbar}\left\{\frac{1}{\mathcal{T}}, \{\{\mathcal{F}^c{}_a\mathcal{F}^d{}_b V\mathcal{M}^b, \pi_c\}, \pi_d\}\right\}
\end{aligned}
$$

$$\Omega^a_{(2)} = \frac{c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left\{\{\pi_e, \mathcal{F}^e{}_b\}, \left\{\pi_f, \left[\epsilon^{abc}\left(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\} + \frac{c}{2}\Xi^a$$

$$+ \frac{ec^2}{4}\epsilon^{abc}\left\{\frac{1}{\mathcal{T}}, \left(\{\mathcal{F}^d{}_b, \pi_d\}V^2\mathfrak{E}_c - V^2\mathfrak{E}_b\{\mathcal{F}^d{}_c, \pi_d\}\right)\right\}$$

$$- \frac{c}{4\hbar}\epsilon^{abc}\left\{\frac{1}{\epsilon'}, \left(\{\mathcal{F}^d{}_b, \pi_d\}V\mathcal{P}_c - V\mathcal{P}_b\{\mathcal{F}^d{}_c, \pi_d\}\right)\right\}.$$

Dirac particle in a gravitational wave with allowance for electromagnetic interactions

$$ds^2 = c^2 dt^2 - \underline{g}_{AB}\, dx^A dx^B - dz^2$$

Here $\underline{g}_{AB} = \underline{g}_{AB}(\varphi)$, with $\varphi = \omega(t - z/c)$.

The FW Hamiltonian reduces to

$$\mathcal{H}_{FW}^{(1)} = \beta\epsilon' + \frac{\hbar c^2}{8}\left\{\frac{1}{\epsilon'}, \epsilon^{cae}\Pi_e\{\pi_b, \mathcal{F}^d{}_c\partial_d\mathcal{F}^b{}_a\}\right\},$$

$$\mathcal{H}_{FW}^{(2)} = \frac{\hbar c}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\Sigma_a\{\pi_e, \mathcal{F}^e{}_b\}, \left\{\pi_f, \epsilon^{abc}\dot{\mathcal{F}}^f{}_c\right\}\right\}\right\},$$

$$\mathcal{H}_{FW}^{(3)} = e\Phi - \frac{e\hbar c^2}{4}\left\{\frac{1}{\epsilon'}, \Pi^a\mathfrak{B}_a\right\}$$

$$+ \frac{e\hbar c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left[\Sigma_a\epsilon^{abc}\left(\{\mathcal{F}^d{}_b, \pi_d\}\mathfrak{E}_c - \mathfrak{E}_b\{\mathcal{F}^d{}_c, \pi_d\}\right) - 2\hbar\mathcal{F}^b{}_a\partial_b(\mathfrak{E}^a)\right]\right\},$$

$$\mathcal{H}^{(4)}_{FW} = -\frac{c}{8}\left\{\frac{1}{\epsilon'}, \left[\Sigma_a \epsilon^{abc}\left(\{\mathcal{F}^d{}_b, \pi_d\}\mathcal{P}_c - \mathcal{P}_b\{\mathcal{F}^d{}_c, \pi_d\}\right) - 2\hbar\mathcal{F}^b{}_a\partial_b(\mathcal{P}^a)\right]\right\} - \Pi^a\mathcal{M}_a$$

$$+\frac{c^2}{4}\left\{\frac{1}{\mathcal{T}}, \left(\Pi^a\{\{\mathcal{F}^c{}_a\mathcal{F}^d{}_b\mathcal{M}^b, \pi_c\}, \pi_d\} + \beta\hbar\left\{\mathcal{F}^b{}_a\mathcal{J}^a, \pi_b\right\}\right)\right\},$$

$$\epsilon' = \sqrt{m^2c^4 + \frac{c^2}{4}\delta^{ac}\{\pi_b, \mathcal{F}^b{}_a\}\{\pi_d, \mathcal{F}^d{}_c\}}, \qquad \mathcal{T} = 2\epsilon'(\epsilon' + mc^2),$$

$$\mathcal{J}^a = e^{abc}\mathcal{F}^d{}_b\partial_d(\mathcal{M}_c) + \frac{\partial\mathcal{P}^a}{c\partial t}, \qquad \mathcal{F}^b{}_a = W^b{}_{\widehat{a}}.$$

The spin motion is defined by

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}_{(2)} \times \mathbf{\Pi}.$$

$$\Omega^a_{(1)} = \frac{c^2}{4}\left\{\frac{1}{\epsilon'}, \{\pi_e, \epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c\}\right\}$$
$$- \frac{ec^2}{2}\left\{\frac{1}{\epsilon'}, \mathfrak{B}_a\right\} - \frac{2}{\hbar}\mathcal{M}_a$$
$$+ \frac{c^2}{2\hbar}\left\{\frac{1}{\mathcal{T}}, \{\{\mathcal{F}^c{}_a\mathcal{F}^d{}_b\mathcal{M}^b, \pi_c\}, \pi_d\}\right\}$$
$$\Omega^a_{(2)} = \frac{c}{8}\left\{\frac{1}{\mathcal{T}}, \left\{\{\pi_e, \mathcal{F}^e{}_b\}, \left\{\pi_f, \epsilon^{abc}\dot{\mathcal{F}}^f{}_c\right\}\right\}\right\}$$
$$+ \frac{ec^2}{4}\epsilon^{abc}\left\{\frac{1}{\mathcal{T}}, \left(\{\mathcal{F}^d{}_b, \pi_d\}\mathfrak{E}_c - \mathfrak{E}_b\{\mathcal{F}^d{}_c, \pi_d\}\right)\right\}$$
$$- \frac{c}{4\hbar}\epsilon^{abc}\left\{\frac{1}{\epsilon'}, \left(\{\mathcal{F}^d{}_b, \pi_d\}\mathcal{P}_c - \mathcal{P}_b\{\mathcal{F}^d{}_c, \pi_d\}\right)\right\}.$$

A search for new observable effects can be performed

Summary

- Relativistic Foldy-Wouthuysen transformation has been carried out for a relativistic Dirac particle in strong gravitational and electromagnetic fields and operator and semiclassical equations of spin motion have been derived
- The results obtained can probably find astrophysical applications. The relativistic Lense-Thirring effect has been analyzed and bounds on the Cartan torsion have been obtained
- Spin interactions with electromagnetic and gravitational fields are similar
- General description of a Dirac particle in strong gravitational and electromagnetic has been fulfilled. As an example, a field of a gravitational wave has been considered

Thank you for your attention