National Academy of Sciences
Joint Institute for Power and Nuclear Research - Sosny

Chaos Assisted Tunneling. Instanton Approach

V.KUVSHINOV, A.KUZMIN

JIPNR – Sosny NAS Belarus, Minsk

e'mail: kuvshinov2003@gmail.com

NONLINEAR PHENOMENA

IN COMPLEX SYSTEMS (XXIY NPCS) 2017

May 16-19, 2017

Minsk
2017

Content

- Chaos and Gauge Field Theories
- Chaos assisted tunneling (CAT)
- Approaches to chaos assisted tunneling (CAT)
- Approach based on instanton techniques
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Chaos and Gauge Field Theories

- **Classical gauge fields demonstrate chaotic behavior** (Savidi, PL,1977; Kawabe, PRD,1990; Kuvshinov,NPCS,1999)

- **Intermittency and chaos in multiple and branching processes of strong interactions (QCD) , Jets, QGP** (Kittel, Dremin,…, Kuvshinov…seventies, eighties)

- **Decoherence, Confinement of colour, instability,** squeezing, entanglement, decreasing of Purity, Fidelity, Information in Stochastic QCD Vacuum as environment (Simonov,UPN, 1996; Kuvshinov, 2003-1016,…)

- **Mathematical apparatus of gauge field theories (namely QCD) can be used in chaos theory and vice versa. V. Kuvshinov, A. Kuzmin. Gauge Fields and Theory of Determenistic Chaos (Belorussian Science, Minsk, 2006, p. 1-268 in Russian).**
- **In particular:**

  Instanton technique can be applied to investigate chaos assisted tunneling regime (Chaos assisted instanton tunneling (chaotic instanton solutions, dilute instanton gas squeezing, exponential widening of the ground quasi-energy zone, numerical simulations) (Kuvshinov, A.Kuzmin et al, PR, APP, 2003-2006)

- **Mathematical apparatus of the gauge field theory can be applied to investigate the stability of holonomic quantum computations** (The influence of the classical control errors on holonomic quantum computations. Fidelity of HQC, Wilson loop and non-Abelian Stokes theorem, robust Hadamard gate for HQC)

Chaos assisted tunneling (CAT)

Despite its genuine quantal character, dynamical tunneling is strongly sensitive to details of the **underlying classical phase space** [Creagh, in Tunneling in Complex Systems, edited by S. Tomsovic (World Scientific, Singapore, 1998)]

Chaos in classical system gives acceleration (Lin, Ballentine, PRL, 1990) or slowing down (Grossman et al, PRL, 1991) the process of tunneling up to several order of magnitude = CAT

(Quantum tunneling and chaos in a driven anharmonic oscillator. The Husimi distribution is computed for a particle in a double-well potential and an oscillatory driving force. The extended phase space of the classical system contains two disjoint stable tubes of regular orbits, embedded in a chaotic sea. For the quantum system we find coherent oscillatory tunneling between these stability tubes, at a rate many orders of magnitude greater than the rate of ordinary undriven tunneling).

- **Both phenomena are seen at experiments**: **C. Dembowski et al., Phys. Rev. Lett. 84, 867 (2000), D. A. Steck et al., Science 293, 274 (2001), W. K. Hensinger et al., Nature 412, 52 (2001).**
- **Approaches to chaos assisted tunneling (CAT)**
  - Numerical methods based on Floquet theory.
  - Three-level model for chaos assisted tunneling.
  - Path integral approach for billiard systems.
  - Quantum mechanical amplitudes in complex configuration space.
  - **Approach based on instanton technique**

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Andrea Addazi Chaotic instantons in scalar field theory arXiv:1607.08360v1 [hep-th] 28 Jul 2016

Approach based on instanton techniques

An **instanton** is a -solution of equations of motion with a finite, non-zero action, either in quantum mechanics or in quantum field theory. More precisely, it is a solution of the equations of motion of the classical field theory on a Euclidean spacetime

Chaotic instanton (CI).

- **CHAOS ENFORCED INSTANTON TUNNELLING IN ONE-DIMENSIONAL MODEL WITH PERIODIC POTENTIAL** (Kuvshinov, Kuzmin, PR, 2003)

- (The influence of chaos on properties of dilute instanton gas in quantum mechanics.is studied. We demonstrate on the example of one-dimensional periodic potential that small perturbation leading to chaos squeezes instanton gas and increases the rate of instanton tunnelling).

- Chaotic instanton is the solution of the Euclidean equations of motion of the perturbed system. This configuration is responsible for the enhancement of tunneling

- Hamiltonian of the system under consideration is taken in the form

$$\overline{H} = \frac{1}{2}\overline{p}^2 + \omega_0^2 \cos x - \varepsilon x \sum_{n=-\infty}^{+\infty} \delta(t - n\overline{T})$$

$\overline{T}$ is the real time period of perturbation

The systems with spatially periodic **potential are well-studied in solid-state physics (Kittel, 1959)and instanton physics (Rajaraman,1982) Perturbation** used in was widely exploited in the systems exhibiting quantum chaos (Berman and, Zaslavsky,1982)

For applying instanton technique we have to consider classical solutions of Hamilton equations in imaginary (Euclidian) time. Hamiltonian has the same form (translated on π) in Euclidian time as in real one. In Euclidian time classical Hamiltonian of the system looks as follows $H = H_0 + V$

$$H_0 = (1/2)p^2 - (\omega_0)^2 \cos x, \qquad V = \varepsilon x \sum_{n=-\infty}^{+\infty} \delta(t - nT).$$

Here H_0 is non-perturbed Hamiltonian of the system and V is the Euclidian potential of the perturbation. In these expressions variables without tilde denote corresponding quantities in imaginary time.

Instanton gas in non-perturbed system

Let us consider at first non-perturbed Hamiltonian , i.e. put ε = 0. The system is characterized by degenerated vacuum structure on the classical level:

$$x_n^{vac}(t) \equiv |n>= \pi n, \quad n = \pm 1, \pm 3, ...$$

As is well-known any tunnelling transition in this system can be represented in terms of path integral in imaginary time:

$$< m|e^{-H_0 \Gamma}|n >= \int [Dx] e^{-S[x]},$$

where Γ is a time of the transition between wells, S[x] denotes **Euclidean action**. The main contribution to is given by instantons. One-instanton configurations are classical solutions of Euclidean equations of motion and describe tunnelling between neighboring vacua. They can be easy found as well as one-instanton action (Rajaraman)

$$x^{inst}(\tau, \tau_0) = \pm \left(arctg \left[e^{\omega_0 (\tau - \tau_0)} \right] - \pi \right), \; S[x^{inst}(\tau, \tau_0)] \equiv S^{inst} = 8\omega_0,$$

where arbitrary parameter τ_0 is a center of the instanton; sings '+' and '−' correspond to instantons and anti-instantons. Multi-instantons are not exact classical solutions, but they give leading contribution to the amplitude of tunnelling between distant wells

$$x^{(n)}(\tau) = \sum_{i=1}^{n} x^{inst}(\tau, \tau_i), \qquad S[x^{(n)}(\tau)] = n S^{inst}.$$

Here we suppose that time intervals between centers of single instantons τ_i are not too close to each other (dilute instanton gas approximation)

Instanton gas in non-perturbed system (2)

➢Let us focus our attention on the amplitude Aq(Γ) of q one-instanton transitions during time interval Γ. The difference between instantons and anti-instantons is not essential in these calculations. Using standard instanton technique (Rajaraman) it is easy to obtain Poisson distribution on q for the amplitude in Gauss approximation:

$$A_q(\Gamma) = N\frac{1}{q!}e^{-qS^{inst}}\left(\Gamma\sqrt{S^{inst}}\right)^q,$$

where factor N provides correct normalization, $A_0(\Gamma)$ means absence of instantons. The average number of instantons for the time Γ is $< q >= e^{-S^{inst}}\Gamma\sqrt{S^{inst}}.$

➢Thus in imaginary time we have a gas consisted of instantons with average time interval between them η_0 and average density ρ_0:

$$\eta_0 = \frac{\Gamma}{< q >} = \frac{e^{S^{inst}}}{\sqrt{S^{inst}}}, \qquad \rho_0 = \eta_0^{-1} = e^{-S^{inst}}\sqrt{S^{inst}} = 2\sqrt{2\omega_0}e^{-8\omega_0}.$$

➢Let us suppose large value of one-instanton action . It corresponds to **high energy barriers**. In this case we obtain strongly **rarefied instanton gas**. It should be noted that result was obtained by using of both exact and approximate classical solutions.

Taking into account only exact solutions leads to zero instanton density ρ_0 (or infinite interval η_0).

Finite density can appear in non-perturbed system only due to approximate solutions contribution.

Squeezing of instanton gas due to small perturbation

➢ **Now we consider the case with ε ≠ 0 and estimate an average interval between instanton transitions (inverse density of instantons in instanton gas) for the perturbed system. For this purpose we represent the perturbation in the form**

$$V = \frac{\epsilon}{T} x \left(2 \sum_{m=-\infty}^{+\infty} \cos(m\nu\tau) + 1 \right).$$

here v ≡ 2π/T

Perturbation destroys separatrix of non-perturbed system and on its place stochastic layer appears Lichtenberg, . Lieberman We primarily estimate the width of stochastic layer. We use method reviewed in Zaslavsky. Exact equation of motion for action variable is

$$\dot{I} = \frac{dI}{dH_0} \dot{H}_0 = -\frac{\epsilon}{\omega T} \dot{x} \left(2 \sum_{m=0}^{+\infty} \cos(m\nu\tau) + 1 \right),$$

Here I denotes action variable and ω ≡ dH0/dI is a frequency of nonlinear oscillations (see Lichtenberg, Lieberman, Consider behavior of the system near separatrix (in imaginary time). Dependence of velocity ẋ on time has the form of rare soliton-like impulses. Each impulse corresponds to rapid transition between two neighbor peaks of potential. Long time interval between two impulses is the time to get over a peak. Action variable changes mainly during the impulse of velocity. Introduce phase of external force ϕ defined as ˙ϕ = v. To study chaotic behavior of the system we transform differential equation C. Callan, Dashen

$$\overline{I} = I + \frac{\epsilon}{T\omega(I)} C(I, \varphi) \qquad \overline{\varphi} = \varphi + \frac{\pi\nu}{\omega(\overline{I})},$$

where (I, ϕ) denote values of action variable and phase just after the impulse of velocity, (I, ϕ) are the same quantities after previous impulse and

$$C(I, \varphi) = -\int_{\Delta\tau} d\tau \dot{x} \left(2 \sum_{m=0}^{+\infty} \cos(m\varphi(t)) + 1 \right).$$

Here we integrate over time interval of velocity's impulse τ . For small ε

$$\overline{\varphi} \simeq \varphi + \frac{\pi\nu}{\omega(I)} - \frac{\epsilon\nu^2}{2\omega^3} \frac{d\omega}{dI} C(I, \varphi).$$

Therefore parameter of local instability Zaslavsky is $K = \left| \frac{\delta\overline{\varphi}}{\delta\varphi} - 1 \right| = \frac{\pi\epsilon\nu}{T\omega^3} \left| \frac{d\omega}{dI} \right| C_0, \quad C_0 \equiv \left| \frac{\partial C}{\partial \varphi} \right|$

Here Hs ≡ (ω0)2 is the energy of non-perturbed system on the separatrix.

Squeezing of instanton gas due to small perturbation (2)

Condition K ≥ 1 means that dynamics of the system is locally unstable. Local instability leads to mixing and chaos Zaslavsky, Phys.Rept. 80, 157 (1981);Thus condition K = 1 gives the estimation for the width of the stochastic layer as follows

$$|H_b - H_s| \approx \frac{4\epsilon\nu^2}{\omega_0} e^{-\pi\nu/2\omega_0},$$

under the assumption that v > ω0. Here Hb is an estimated value of energy on boundary of the stochastic layer. A set of trajectories appearing in the stochastic layer can be considered as new multi-instanton-like exact solutions of Euclidian equations of motion for perturbed system. These solutions demonstrate finite intervals between instanton transitions. The reason is that for majority of trajectories in stochastic layer finite time is needed to pass from one potential well to another. Thus density of instanton gas strongly increases in comparison with density at ε= 0. Average time interval between instantons in the instanton gas for pertu

$$\eta \approx \frac{\pi}{\omega_{av}} \qquad \omega(H) \equiv \frac{dH_0}{dI} \approx \frac{\pi}{\sqrt{2}} \frac{\sqrt{H_s + H}}{\ln \frac{16(H_s+H)}{H-H_s}}, \quad |H - H_s| \ll H_s.$$

Here $\omega_{av} \equiv \omega([H_b + H_s]/2)$. Near the separatrix following approximation for the frequency can be used [11]

$$\omega(H) \equiv \frac{dH_0}{dI} \approx \frac{\pi}{\sqrt{2}} \frac{\sqrt{H_s + H}}{\ln \frac{16(H_s+H)}{H-H_s}}, \quad |H - H_s| \ll H_s. \tag{21}$$

Thus interval between instanton transitions is the next

$$\eta \approx \frac{\pi\nu}{2\omega_0^2} + \frac{1}{\omega_0} \ln \frac{8\omega_0^3}{\epsilon\nu^2},$$

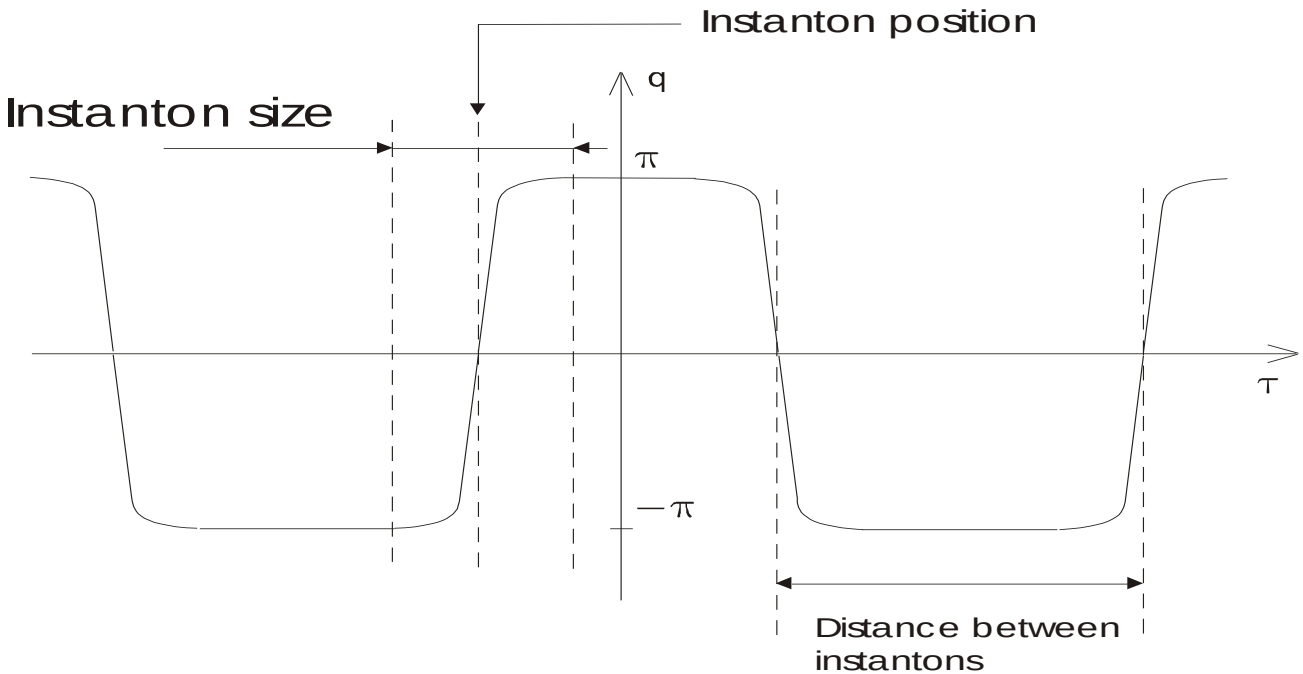
and the density of instantons in instanton gas for the perturbed system can be estimated as follows

$$\rho = \frac{1}{\eta} = \left(\frac{\pi\nu}{2\omega_0^2} + \frac{1}{\omega_0} \ln \frac{8\omega_0^3}{\epsilon\nu^2} \right)^{-1}. \quad \frac{\rho}{\rho_0} \approx \frac{1}{\sqrt{2}} \frac{\omega_0 \sqrt{\omega_0} e^{8\omega_0}}{\pi\nu + 2\omega_0 \ln \frac{8\omega_0^3}{\epsilon\nu^2}}.$$

This ratio is large at large enough one-instanton action. and small but nonzero ε.Thus small perturbation can strongly increase the density of instanton gas. While to get non-zero value of ρ0 we had to consider approximate solutions. Thus the limit ε → 0 can not be applied directly in (ρ/ρ0Nevertheless we have correspondence between results obtained for perturbed and non-perturbed systems in this limit. Namely, ρ (see (23)) at ε = 0 and ρ0 are equal to zero if we take into account only exact solutions of Euclidian equations of motion for both cases. Let us apply now our formal computation to the physical phenomena in real time. Classical instanton solutions in imaginary time describe quantum tunneling transitions in real time. Some observables can be directly expressed through the instanton density ρ Rajaraman. In particular, rate of the tunnelling between neighbour potential wells (the number of tunnelling transitions per unit of time) and probability of tunnelling are proportional to squared density of instantons . Spectrum and width of the lowest energy zone E read

$$E_\theta \approx \frac{1}{2}\omega_0 - 2\rho\cos\theta, \quad 0 \le \theta \le \pi, \qquad \Delta E \approx 4\rho.$$

Dilute instanton gas squeezing



- Average density of the non-perturbed instanton gas:

$$\rho_0 = 2\sqrt{2\omega_0}\,\exp(-8\omega_0)$$

- The ratio of the average perturbed instanton gas density to ρ_0:

$$\frac{\rho}{\rho_0} = \frac{1}{\sqrt{2}}\,\frac{\omega_0\sqrt{\omega_0}}{\pi\nu + 2\omega_0 \ln\dfrac{8\omega_0^3}{\varepsilon\nu^2}}\exp(8\omega_0)$$

Exponential widening of the ground quasienergy zone.

- **Dynamical tunneling amplitude with the contribution of the chaotic instanton solutions**:

$$A = N\int_0^{\Delta H} d\xi \int_{-\infty}^{+\infty} dc_0 \sqrt{S[x^{ins}(\tau,\xi)]}\exp(-S[x^{ins}(\tau,\xi)]) \approx N\sqrt{8\omega_0}\,\Gamma e^{-8\omega_0}\exp\left(\frac{\pi\Delta H}{\omega_0}\right)$$

Euclidean action of chaotic instanton solution:

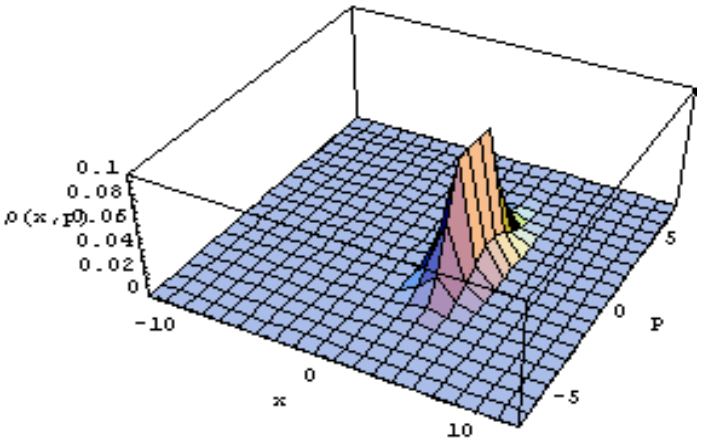
$$S[x^{ins}(\tau,\xi)] \approx 8\omega_0 - \frac{\pi\xi}{\omega_0}$$

- Ground quasienergy zone width:

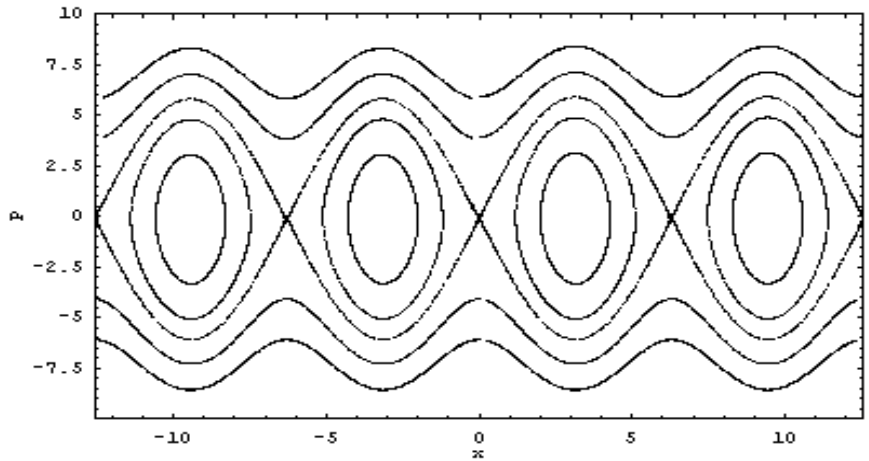
$$\Delta E \approx N e^{-8\omega_0}\sqrt{8\omega_0}\exp\left(\frac{\pi\Delta H}{\omega_0}\right)$$

Numerical simulation of dynamical tunneling properties

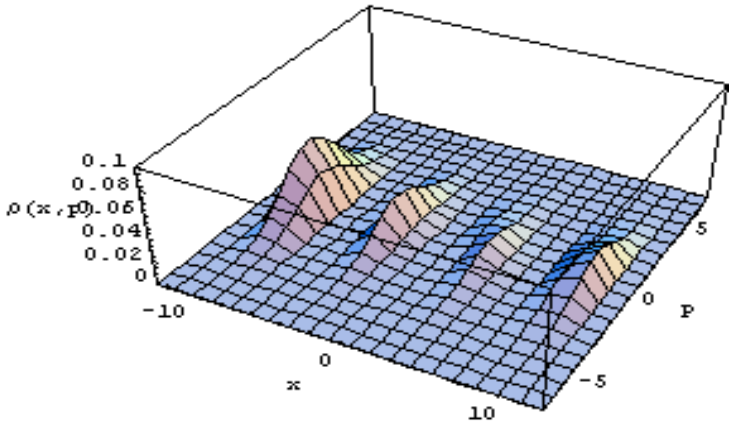
Initial wave packet:



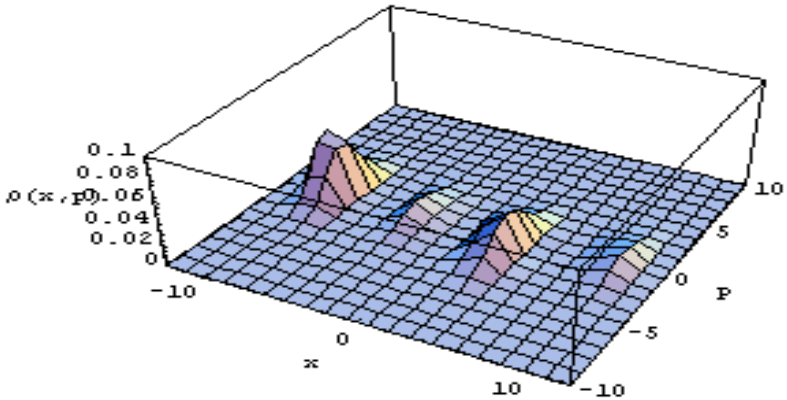
Phase portrait of the non-perturbed system:



Wave packet after the time interval T with no perturbation:

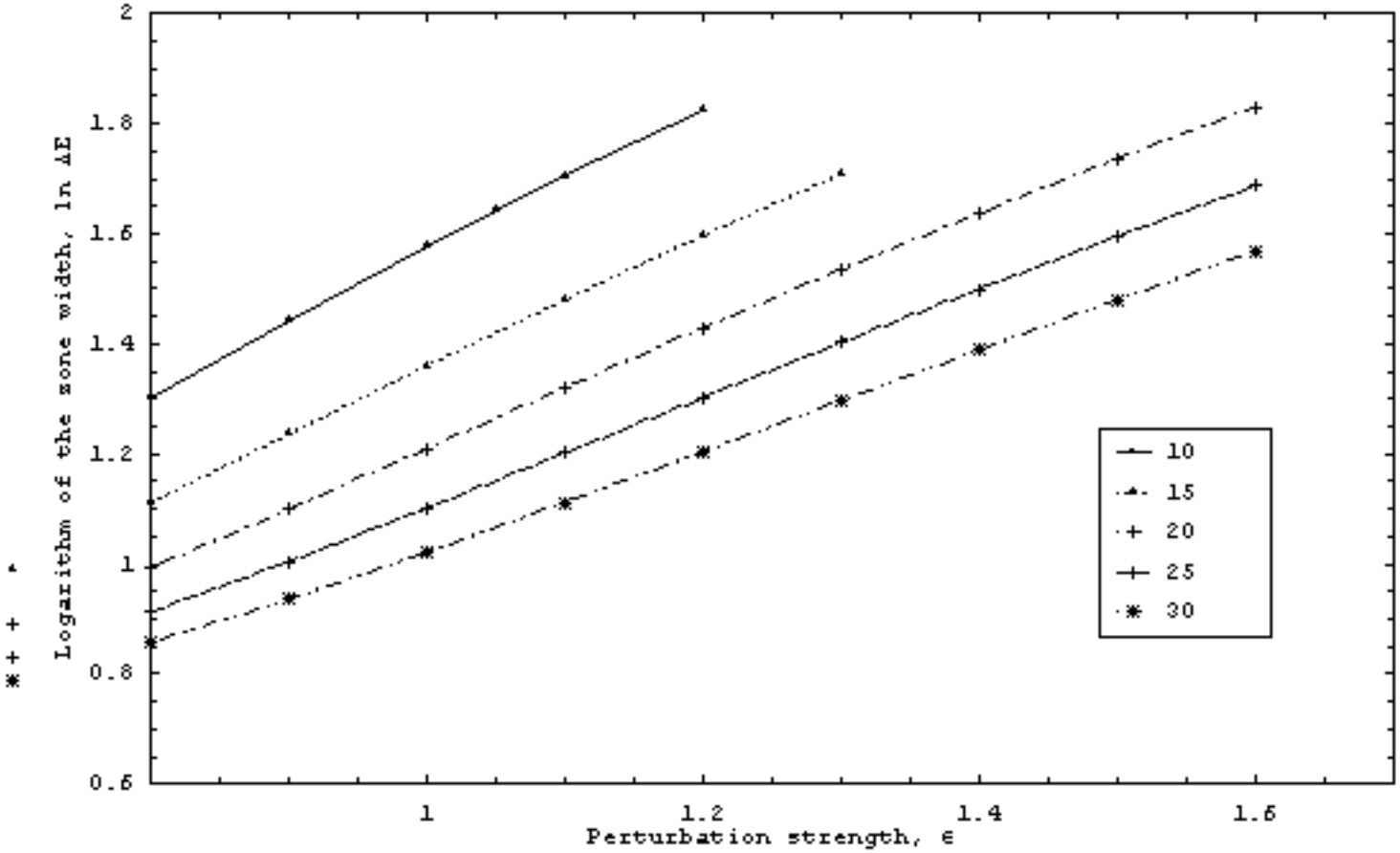


Wave packet after the time interval T/13.7 when the perturbation acts:



Process of tunneling accelerates by about the order of magnitude (as in analytic calculation)

Exponential widening of the ground quasienergy zone (numerical results)



Log quasienergy splitting as a function of the strength of the perturbation for di®erent values of model parameters ***Lines - analytical formula , points - numerical results***

Discussion

- Thus squeezing of the instanton gas and increase of the density ρ in imaginary time mean that small perturbation leading to chaos can essentially enhance the tunnelling rate and lead to the widening of the energy zone in comparison with non-perturbed system
- Both these results are consequences of the perturbation leading to destruction of the single non-perturbed instanton solution and appearance of manyfold of chaotic perturbed instantons.
- An increase of the number of instanton solutions provides a larger number of variants for particle to reach one vacuum from another that results in the increase of the rate of tunnelling.
- The decrease of the life-time of the particle in the certain vacuum of the system means the widening of the energy zone
- On the other hand our results can be considered as a demonstration on the simple model of application of the instanton method to the problem of chaos assisted tunnelling.
- We have demonstrated on the example of one-dimensional periodic potentialt hat small perturbation leading to chaotic behavior of the system strongly influences on the properties of instanton gas.
- Our estimations show that classical chaos can greatly increase the density of instanton gas and rate of instanton tunnelling.
- Both instanton solutions and chaotic behavior can exist in complex systems like field theories. The relation between chaos and instantons in such theories is not trivial. Theory of strong interactions (QCD) is the most interesting example, where the investigation of instanton gas (or instanton liquid) could shed light on the structure of hadrons

Conclusion

- Mathematical apparatus of QCD, namely chaotic instantons, can be effectively used to describe some properties of the chaos assisted tunneling regime –
- squeezing of the dilute instanton gas
- exponential widening of the ground quasi-energy zone
- description of CAT phenomenon

- Mathematical apparatus of QCD primarily developed in the framework of the confinement theory (QCD) is useful for the investigation of the stability of holonomic quantum computations

- We would like to emphasize the mathematical analogy between Wilson loop and fidelity in QCD and fidelity of holonomic quantum computations.

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Thank you for the attention!