Chapter 1

Refraction and mirror reflection from a matter–vacuum plane boundary

1.1 Radiation and Scattering of Waves and Particles

The variety of physical phenomena and processes occurring around and inside us on a macro and micro scale can really confuse anyone who takes first steps, trying to understand and describe the laws they obey. Nevertheless, this state of confusion disappears when, after some effort, numerous relations and analogies between processes and phenomena, which sometimes seem utterly diverse, suddenly become unveiled. As a result of such unity, the apparently different research methods, which we use to explore the world around us, appear to be essentially the same.

The oldest method of studying physical phenomena is visual observation. This method of exploring the world seems to have nothing in common with the investigation of the structure of elementary particles by bombarding nuclei with protons accelerated to very high energies.

However, in both cases we deal with inherently the same method, actually, we conduct one and the same experiment. In both cases we have 1) a particle (wave) source – either the Sun lighting objects or an accelerator generating energetic protons; 2) a target – either the illuminated objects or the bombarded nuclei; 3) a detector – the human eye and brain in one case or a proportional chamber (photomultiplier and so on) with a data processing system in the other (and finally – the human brain again, the brain of the observer).

Scattering of particles (neutrons, γ-quanta, photons and others) by atoms, molecules, and nuclei is widely used for studying physical phenomena.

A typical experiment based on particle (wave) scattering is performed as follows (Fig. 1.1)

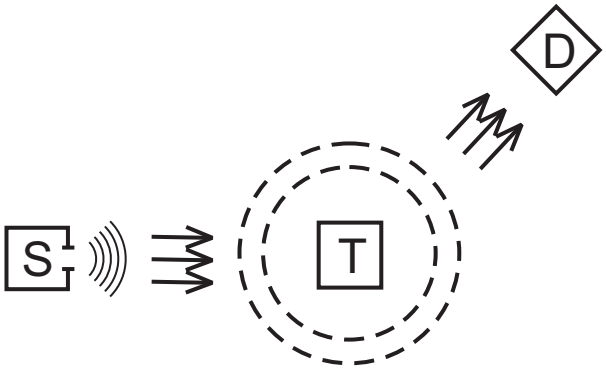

Fig. 1.1 Scattering of a primary wave by the target

Particles generated by source S fall on target T, interact with the target atoms (nuclei) and, as a result, change the direction of motion and produce new particles. Detector D situated at an angle with the incident direction of the primary beam detects deflected (scattered) particles or particles produced through collision.

To describe scattering properties of atoms, it is convenient to introduce the quantity $d\sigma/d\Omega$, called differential scattering cross section per unit solid angle Ω (see, for example, [Goldberger and Watson (1984); Landau and Lifshitz (1975, 1977)]):

$$d\sigma/d\Omega = j_{\rm scat}^2 r^2/j_0\,, \tag{1.1}$$

where $j_{\rm scat}$ is the flow of particles scattered by the atom (nucleus) in the direction of the detector; j_0 is the initial flux of particles incident on the target; r is the distance between the scattering nucleus and the detector. In the polar coordinate system, $d\Omega = \sin\vartheta\, d\vartheta\, d\varphi$, where ϑ and φ are the polar and azimuth angles determining the observation direction. As the values of j_0 and $j_{\rm scat}$ can be measured experimentally, $d\sigma/d\Omega$ can also be determined.

The question naturally arises of what information about the processes taking place in the target (scatterer) can be obtained in such an experiment, and whether it is worth trying to measure $d\sigma/d\Omega$.

To answer this question, let us first take into consideration that due to inherent particle–wave dualism of the quantum–mechanical description of the behavior of micro-objects, the peculiarities of the interactions between microparticles may be revealed in terms of pure classical wave description of these phenomena.

In the beginning, let us consider, by way of example, the phenomenon of generating and scattering of γ-quanta by nuclei as the process of generating and scattering of electromagnetic waves by a system of charges.

First analyze the work of the source. From the classical viewpoint, reactions proceeding in the source result in the formation of charges moving along certain trajectories, which are determined by the forces acting within the system under consideration. Suppose that the charge motion is non-relativistic. Assume also that each particle has a charge e and vibrates about the equilibrium position at point $\vec{R}_n$. Let a characteristic vibration period be long in comparison with the time required for a photon to pass the distance of the order of the vibration amplitude; the characteristic wavelength emitted by the charge is much greater than the amplitude of charge vibration. In this case the electric field strength $\vec{E}_n$ in the emitted wave at time t at a distance longer than the wavelength is described as follows (see e.g. [Landau and Lifshitz (1975)]):

$$\vec{E}_n = \frac{1}{c^2|\vec{r}-\vec{R}_n|}\left[\vec{n}_n\left(\vec{n}\ddot{\vec{d}}_n\left(t-\frac{|\vec{r}-\vec{R}_n|}{c}\right)\right)-\ddot{\vec{d}}_n\left(t-\frac{|\vec{r}-\vec{R}_n|}{c}\right)\right] \times\theta\left(t-t_n-\frac{|\vec{r}-\vec{R}_n|}{c}\right), \tag{1.2}$$

where

$$\vec{d}_n\left(t-\frac{|\vec{r}-\vec{R}_n|}{c}\right)=e\vec{\eta}_n\left(t-\frac{|\vec{r}-\vec{R}_n|}{c}\right)$$

is the electric dipole moment produced by particle n with respect to the equilibrium point $\vec{R}_n$ at time $t-|r-\vec{R}_n|/c$; $\vec{\eta}_n$ is the position vector of charge n with respect to point $\vec{R}_n$; t_n is the starting time of charge vibration; $\theta(x)$ is the Heaviside unit function:

$$\theta(x)=\begin{cases}1 \text{ when} & x\geq 0\,,\\ 0 \text{ when} & x<0\,.\end{cases}$$

The function θ in Eq. (1.2) indicates that a nonzero value of the electric field strength $\vec{E}_n$ in a wave appears at the observation point only at times t greater than the time required for the photon to pass the distance between the source and the observation point.

To describe the charge motion, we adopt the damped oscillator model:

$$\ddot{\vec{\eta}}_n+\Gamma_n\dot{\vec{\eta}}_n+\omega_n^2\vec{\eta}_n=0\,. \tag{1.3}$$

Let at time moment t_n the oscillator have the coordinate $\vec{\eta}_{n0}$ and velocity $\vec{v}_{n0}$. Suppose that $\omega_n \gg \Gamma_n$. In this case the solution of Eq. (1.3) has the form

$$\vec{\eta}_n = \left(\vec{c}_{1n} e^{i\omega_n (t-t_n)} + \vec{c}_{2n} e^{-i\omega_n (t-t_n)}\right) \times e^{-\frac{\Gamma_n}{2}(t-t_n)} \theta(t-t_n)\,, \tag{1.4}$$

where

$$\vec{c}_{1n} = \frac{i\omega_n \vec{\eta}_{n0} + \vec{v}_{n0}}{2i\omega_n}\,; \vec{c}_{2n} = \frac{i\omega_n \vec{\eta}_{n0} - \vec{v}_{n0}}{2i\omega_n}\,.$$

Provided that $\omega_n \gg \Gamma_n$, from Eq. (1.3) we immediately obtain for $\ddot{\vec{\eta}}_n$:

$$\ddot{\vec{\eta}}_n \approx -\omega_n^2 \vec{\eta}_n\,. \tag{1.5}$$

Substitution of Eq. (1.4)–Eq. (1.5) into Eq. (1.2) gives the following expression for the field induced by the oscillator:

$$\begin{aligned} \vec{E}_n = & \left\{ \vec{a}_{1n} \frac{\exp[i\omega_n (t - t_n - \frac{|\vec{r}-\vec{R}_n|}{c})]}{|\vec{r}-\vec{R}_n|} \right. \\ & \left. + \vec{a}_{2n} \frac{\exp[-i\omega_n (t - t_n - \frac{|\vec{r}-\vec{R}_n|}{c})]}{|\vec{r}-\vec{R}_n|} \right\} \\ & \times \exp[-\frac{\Gamma_n}{2}(t - t_n - \frac{|\vec{r}-\vec{R}_n|}{c})] \theta\left(t - t_n - \frac{|\vec{r}-\vec{R}_n|}{c}\right)\,, \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} \vec{a}_{1n} &= -\frac{e\omega_n^2}{c^2}[\vec{n}_n(\vec{n}_n \vec{c}_{1n}) - \vec{c}_{1n}]\,; \\ \vec{a}_{2n} &= -\frac{e\omega_n^2}{c^2}[\vec{n}_n(\vec{n}_n \vec{c}_{2n}) - \vec{c}_{2n}]\,. \end{aligned} \tag{1.7}$$

According to Eq. (1.6), the damped oscillator generates spherical damped waves, whose phase and amplitude depend on the oscillator coordinate $\vec{R}_n$ and the starting moment t_n of charge motion. The field $\vec{E}(\vec{r}, t)$ induced by the system of charges is a superposition of fields induced by particular charges, i.e.,

$$\vec{E}(\vec{r},\, t) = \sum_n \vec{E}_n(\vec{r},\, t). \tag{1.8}$$

Let us thoroughly examine Eq. (1.6)–Eq. (1.8). In the most common case, the radiating atom (nucleus) appears in a source as a result of some stochastic process (electron bombardment of atoms upon gas discharge, α- or β-

decay of nuclei, and so on). In consequence, the amplitudes $\vec{a}_{1(2)n}$, the coordinates $\vec{R}_n$, and the vibration starting times t_n become random values distributed according to a certain law. The experimentally measured value of the field in Eq. (1.8) or its degrees should be averaged over these distributions.

According to Eq. (1.1), to determine $d\sigma/d\Omega$, one needs to know the particle flux (radiation intensity). The intensity J of electromagnetic radiation is determined by the Poynting vector, i.e.,

$$J(\vec{r},t) \sim \vec{E}^2(\vec{r},t) = \left(\sum_n \vec{E}_n(\vec{r},t)\right)^2$$
$$= \sum_n \vec{E}_n^2(\vec{r},t) + \sum_{n\neq n'} \vec{E}_n(\vec{r},t)\vec{E}_{n'}(\vec{r},t)\,. \tag{1.9}$$

The average value of intensity taken over the distribution of $\vec{a}_{1(2)}$, $\vec{R}_n$, and t_n is

$$\langle J(\vec{r},t)\rangle = \langle \sum_n \vec{E}_n^2(\vec{r},t)\rangle + \langle \sum_{n\neq n'} \vec{E}_n(\vec{r},t)\vec{E}_{n'}(\vec{r},t)\rangle\,. \tag{1.10}$$

When there are no correlations among the values of $\vec{a}_{1(2)n}$, $\vec{R}_n$, and t_n, the second term in Eq. (1.10) associated with interference of the fields produced by different sources is equal to zero; the radiation intensity of the source is the sum of radiation intensities of individual atoms (nuclei).

Under conventional conditions of particle creation in radioactive substances, reactors, and accelerators, the second term is zero with high accuracy. From Eq. (1.10) is seen that the presence of correlation among the values of $\vec{a}_{1(2)n}$, $\vec{R}_n$, and $\vec{t}_n$ is necessary in order to get a nonzero interference contribution. It is evident, in particular, that the correlations exist and the second term in Eq. (1.10) is nonzero if all the sources have the same initial vibration amplitude and velocity and are periodically distributed and periodically excited with the time period τ: $t_n = n\tau$.

A more complicated source of correlations appears with account of the fact that the field produced by the source located, for example, at point $\vec{R}_{n'}$ forces vibrations of the oscillator located at point $\vec{R}_n$. As a consequence, the equation of motion for the deviation of $\vec{\eta}_n$ takes the form

$$\ddot{\vec{\eta}}_n + \Gamma_n\dot{\vec{\eta}}_n + \omega_n^2\vec{\eta}_n = \sum_{n'\neq n} \vec{g}_{nn'}\,, \tag{1.11}$$

where $\vec{g}_{nn'}$ is the force acting on n-oscillator from the fields produced by oscillators n'.

In this case the solution of Eq. (1.11) is the sum of the general solution of the homogeneous equation Eq. (1.3) and the specific solution of Eq. (1.11).

Forced vibrations cause extra generation of the fields, whose amplitudes and phases depend on the variables characterizing all the oscillators. In this chapter we assume that all the oscillators (atoms, nuclei) radiate independently. For this reason, when examining the processes stimulated by the waves produced by the source in the target, it will suffice to study the interaction between the wave emitted by a particular nucleus (atom) and the target.

Qualitatively, the answer is quite obvious. Upon reaching the target nuclei (atoms), the wave generated by the source excites vibrations of charges–oscillators corresponding to them. These vibrations bring about a time–varying electric dipole moment, which, according to Eq. (1.2), leads to production of radiation. This secondary radiation is just the waves scattered by the target. They are also spherical. But unlike Eq. (1.6), their amplitude and phase are defined not only by the properties of the scatterers, but also by the incident wave. As stated above, this wave can bring about a significant correlation between the amplitudes and phases of the waves produced by particular scatterers.

Recall that according to Eq. (1.1), the scattering cross section is determined by the relation between the flows of scattered and incident waves, and for electromagnetic radiation it is:

$$d\sigma = \frac{\langle dJ \rangle}{\langle S \rangle}\,, \tag{1.12}$$

where $\langle dJ \rangle$ is the energy radiated into solid angle $d\Omega$ in 1 second by the system exposed to the incident wave with the intensity characterized by the Poynting vector $\langle S \rangle$. Consequently, the scattering cross section gives information about the properties of the scatterers.

Owing to the linear character of Maxwell equations and the Schrödinger equation, when examining the interaction between the wave and the scatterers, we can first consider the process of scattering of a monochromatic wave. The general solution can be obtained by summing the solutions for monochromatic waves with the amplitudes determining the weight of each monochromatic component of the incident wave packet. Detailed substantiation of the procedure is given in, for example, [Goldberger and Watson (1984)].

So let a monochromatic plane wave be incident on the oscillator. The electric field strength in the wave is

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\vec{r} - \omega t + \alpha)\,, \tag{1.13}$$

where $\vec{E}_0$ is the wave amplitude, $\vec{E}_0 = E_0\vec{e}$; $\vec{k}$ is the wave phase vector; ω is its frequency; α is the initial phase.

Under the action of $\vec{E}$, the oscillator performs forced vibrations defined by equation:

$$\ddot{\vec{\eta}}_n + \Gamma_n\dot{\vec{\eta}}_n + \omega_n^2\vec{\eta}_n = \frac{e}{m}\vec{E}_0\cos(\vec{k}\vec{R}_n - \omega t + \alpha)\,, \tag{1.14}$$

where e is the charge of the vibrating particle; m is the particle mass.

In writing Eq. (1.14) we assume that the amplitude of forced vibrations is much smaller than the wavelength of the incident radiation, i.e., $k\eta_n \ll 1$. This enables us to replace $\vec{r}$ at the point of charge location by the coordinate of the equilibrium point $\vec{R}_n$ on the right-hand side of Eq. (1.14) in the expression for force. Equation (1.14) is easily solvable and has the form:

$$\begin{aligned}\vec{\eta}_n = {} & \frac{eE_0}{2m}\frac{\vec{e}}{\omega_n^2 - \omega^2 - i\omega\Gamma_n}e^{i(\vec{k}\vec{R}_n - \omega t + \alpha)} \\ & + \frac{eE_0}{2m}\frac{\vec{e}}{\omega_n^2 - \omega^2 + i\omega\Gamma_n}e^{-i(\vec{k}\vec{R}_n - \omega t + \alpha)}\,.\end{aligned} \tag{1.15}$$

According to Eq. (1.2), the radiation field may be represented as follows:

$$\begin{aligned}\vec{E}_n = {} & \frac{eE_0\omega^2}{2mc^2}[n_n[n_n e]]\left\{\frac{\exp[i(\vec{k}\vec{R}_n - \omega t + \alpha)]}{\omega^2 - \omega_n^2 + i\omega\Gamma_n}\right. \\ & \times\frac{\exp[ik|\vec{r} - \vec{R}_n|]}{|\vec{r} - \vec{R}_n|} + \frac{\exp[-i(\vec{k}\vec{R}_n - \omega t + \alpha)]}{\omega^2 - \omega_n^2 - i\omega\Gamma_n} \\ & \left.\times\frac{\exp[-ik|\vec{r} - \vec{R}_n|]}{|\vec{r} - \vec{R}_n|}\right\}\,.\end{aligned} \tag{1.16}$$

Note that the second term in Eq. (1.16) is the complex conjugate of the first term. Therefore the second term will be dropped further on. Moreover, the factor $(E_0/2)\exp[-i(\omega t - \alpha)]$ will also be dropped below as it is common for both the incident and scattered waves.

Thus, the wave field originating in space as a result of scattering of the plane wave has the form:

$$\vec{E}(\vec{r}) = \vec{e}e^{i\vec{k}\vec{r}} + \vec{f}_n\frac{e^{ik|\vec{r} - \vec{R}_n|}}{|\vec{r} - \vec{R}_n|}e^{i\vec{k}\vec{R}_n}\,, \tag{1.17}$$

where

$$\vec{f}_n = \frac{e^2\omega^2}{mc^2}\frac{[\vec{n}_n[\vec{n}_n e]]}{\omega^2 - \omega_n^2 + i\omega\Gamma_n}\,.$$

The amplitude of the diverging spherical wave formed as a result of the action of a unit–amplitude plane wave on the scatterer is called the scattering amplitude. Owing to the vector character of electromagnetic waves, $\vec{f}_n$ is a vector quantity in the case in question. In scattering of scalar waves described, in particular, by the Schrödinger equation, the amplitude f is a scalar. Using the definition of the differential scattering cross section Eq. (1.1)–Eq. (1.12), we obtain that

$$d\sigma/d\Omega = \vec{f}_n^{\,*}\vec{f}_n \tag{1.18}$$

(for scalars $d\sigma/d\Omega = |f|^2$). Particularly, in the considered case we have a well-known expression

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega^2-\omega_n^2)^2+\omega^2\Gamma_n^2}[1-(\vec{n}_n\vec{e})^2]\,. \tag{1.19}$$

Now let a wave be incident on a set of scatterers (oscillators). Under the action of the wave, each oscillator performs forced vibrations, and, as a result, radiates. In other words, the wave is scattered by a set of scattering centers. Due to a linear character of wave equations, the scattered wave is formed by the superposition of the waves emitted by each oscillator, i.e., in this case

$$\vec{E}(\vec{r}) = \vec{e}e^{i\vec{k}\vec{r}} + \sum_n \vec{f}_n \frac{e^{ik|\vec{r}-\vec{R}_n|}}{|\vec{r}-\vec{R}_n|} e^{i\vec{k}\vec{R}_n}\,. \tag{1.20}$$

As in addition to the incident primary wave, each oscillator is affected by the waves emitted by other oscillators, the amplitude $\vec{f}_n$ in the general case is different from the amplitude of scattering by the oscillator Eq. (1.17) in the absence of other centers.

However, this fact will be temporarily neglected. For simplicity, we will next consider scattering of scalar waves. In this case the wave produced in scattering by a set of centers (target) has the form:

$$\Psi(\vec{r}) = e^{i\vec{k}\vec{r}} + \sum_n f_n \frac{e^{ik|\vec{r}-\vec{R}_n|}}{|\vec{r}-\vec{R}_n|} e^{i\vec{k}\vec{R}_n}\,. \tag{1.21}$$

At large distances from the target, Eq. (1.21) takes the form:

$$\lim_{r\to\infty}\Psi(\vec{r}) = e^{ikr} + F\frac{e^{i\vec{k}\vec{r}}}{r}\,, \tag{1.22}$$

where

$$F = \sum_n f_n e^{-i(\vec{k}'-\vec{k})\vec{R}_n}\,, \tag{1.23}$$

$\vec{k}' = k\vec{r}/r$, the origin of coordinates is located inside the target.

It is seen that at large distances from the target a diverging spherical wave with the amplitude F is formed from a set of diverging spherical waves. From Eq. (1.22) follows that the amplitude F of scattering by a few centers is the sum of the amplitudes of scattering by particular centers.

Let us consider the simplest model. Suppose that identical scatterers are continuously distributed over a certain volume with the density $\rho(\vec{r})$ (there may be electrons in the atom, nucleons in the nucleus). Then

$$F(\vec{q}) = f \int e^{-i\vec{q}\vec{r}} \rho(\vec{r}) d^3\vec{r},$$
$$\vec{q} = \vec{k}' - \vec{k}\,. \tag{1.24}$$

According to Eq. (1.24), the scattering amplitude is, in fact, the Fourier transform of the distribution density $\rho(\vec{r})$. As a result, knowing F for all values of the difference $\vec{k}' - \vec{k}$, one can find such an essential characteristic of the scatterer as the density $\rho(\vec{r})$. Though the situation actually often appears to be much more complicated, yet even the foregoing shows the utter importance of knowing the scattering amplitudes. Since the scattering amplitude determines the scattering cross section, experimental determination of the scattering cross section enables one to obtain information about the scattering amplitude, i.e., measuring the scattering cross section is really necessary. However, finding the scattering amplitude from the data referring to the scattering cross section measurements is actually an intricate problem (when measuring $d\vartheta/d\Omega$ we, in fact, determine only $|f|$, which is not sufficient as the scattering amplitude is a complex number). Methods of theoretical calculation of scattering amplitudes are available from, for example, [Goldberger and Watson (1984); Landau and Lifshitz (1977)]. In particular, at scattering of a particle of mass m by a potential $V(\vec{r})$, the scattering amplitude is defined by equation

$$f(\vec{k}', \vec{k}) = -\frac{m}{2\pi\hbar^2} \int e^{-i\vec{k}'\vec{r}} V(\vec{r}) \Psi(\vec{r}) d^3\vec{r}, \tag{1.25}$$

where $\Psi(\vec{r})$ is the wave function satisfying the integral Schrödinger equation of the form

$$\Psi(\vec{r}) = e^{i\vec{k}\vec{r}} - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}') \psi(\vec{r}') d^3\vec{r}'\,. \tag{1.26}$$

Consider a wave forward–scattered (a wave scattered at zero angle) by a set of identical scatterers. In this case $\vec{k}' - \vec{k} = 0$ and [see Eq. (1.23)]

$$F(0) = Nf\,, \tag{1.27}$$

where N is the number of scatterers. As it is seen, $F(0)$ increases with increasing number of scattering centers and at $N \to \infty$, the amplitude of the scattered wave $F(0) \to \infty$. Such a behavior of the amplitude naturally makes us think that Eq. (1.27) is not always suitable.

1.2 Refraction and Mirror Reflection

In this regard, let us scrutinize the process of wave passage through a set of scatterers. A detailed analysis of wave transmission through matter is given in numerous works (see e.g. [Fermi (1950); Lax (1951); Hughes (1954); Goldberger and Watson (1984)].

To avoid awkward mathematical calculations unnecessary for our consideration, we shall treat the process under study within the framework of a simple model.

Let a plane wave $e^{i\vec{k}\vec{r}}$ be incident on matter. In accordance with the above, a divergent spherical wave is produced through the interaction between the wave and the scatterer (atom, nucleus). Suppose for simplicity that the perturbation theory is applicable to the description of the interaction between the wave and a particular scatterer [Landau and Lifshitz (1977); Goldberger and Watson (1984)]. This means that in the first order approximation over the interaction, the wave function $\psi(\vec{r})$ in the integrands of Eq. (1.25) and Eq. (1.26) should be taken as $\psi(\vec{r}) \approx e^{i\vec{k}\vec{r}}$. In this case the wave describing the process of elastic scattering by a single center located at point $\vec{R}_n$ can be written as

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}} + \int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} W(\vec{r}\,' - \vec{R}_n) e^{i\vec{k}\vec{r}\,'} d^3\vec{r}\,' \,, \tag{1.28}$$

where $W(\vec{r}\,' - \vec{R}_n) = -m/2\pi\hbar^2 V(\vec{r}\,' - \vec{R}_n)$; $V(\vec{r}\,' - \vec{R}_n)$ is the interaction potential between the incident particle and the atom (nucleus); m is the particle mass.

At large distances from the scatterer we obtain from Eq. (1.28)

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}} + f(\vec{k}\,', \vec{k}) \frac{e^{ik|\vec{r}-\vec{R}_n|}}{|\vec{r}-\vec{R}_n|} e^{i\vec{k}\vec{R}_n} \,, \tag{1.29}$$

where

$$f(\vec{k}\,', \vec{k}) = -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}\,' - \vec{k})\vec{r}\,'} V(\vec{r}\,') d^3\vec{r}\,'$$

is the amplitude of scattering by the center located at the origin of coordinates, which is calculated in the first–order perturbation theory approximation.

When a particle falls upon a set of scattering centers, the spherical waves formed in the collision with all the atoms (nuclei) should be added to the initial wave $e^{i\vec{k}\vec{r}}$ as it has already been demonstrated.

Consequently,

$$\psi = e^{i\vec{k}\vec{r}} + \sum_n \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} W(\vec{r}' - \vec{R}_n) e^{i\vec{k}\vec{r}'} d^3\vec{r}' \tag{1.30}$$

or

$$\psi = e^{i\vec{k}\vec{r}} + \sum_n \int \frac{e^{ik|\vec{r}-\vec{\eta}-\vec{R}_n|}}{|\vec{r}-\vec{\eta}-\vec{R}_n|} W(\vec{\eta}) e^{i\vec{k}\vec{\eta}} d^3\eta e^{i\vec{r}\vec{R}_n} . \tag{1.31}$$

Suppose that the nuclei are randomly distributed in plane $z = z_0$. A coherent wave formed through the interaction with the plane is obtained by averaging Eq. (1.31) over the position of scatterers in this plane. As a result, we get

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}\vec{r}} + \rho' \int\int \frac{e^{ik|\vec{r}-\vec{\eta}-\vec{R}|}}{|\vec{r}-\vec{\eta}-\vec{R}|} W(\vec{\eta}) e^{i\vec{k}\vec{\eta}} e^{i\vec{k}\vec{R}} d^3\eta d^2R_\perp , \tag{1.32}$$

where $d^2R_\perp$ means integration over the components R_x and R_y in the $z = z_0$ plane; $\vec{R} = (R_x,\, R_y,\, z_0)$; ρ' is the density of the scatterers in the z_0 plane.

Evaluation of the integral over $d^2R_\perp$ gives the following expression for $\langle\psi(\vec{r})\rangle$:

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}\vec{r}} + \frac{2\pi\rho'}{|k_z|} e^{i\vec{k}_\perp\vec{r}_\perp} \int e^{i|k_z||z-\eta_z-z_0|} W(\vec{\eta}) e^{ik_z\eta_z} d^3\eta e^{ik_z z_0} , \tag{1.33}$$

If $z - z_0$ is much larger than the radius of action of the potential $W(\vec{\eta})$ and the observation point is behind the plane, e.i., $z > z_0$,

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}_\perp\vec{r}_\perp} \left(e^{ik_z z} + \frac{2\pi i\rho'}{k_z} \int W(\vec{\eta}) d^3\vec{\eta} e^{ik_z z} \right) , \tag{1.34}$$

i.e.,

$$\psi(\vec{r}) = e^{i\vec{k}_\perp\vec{r}_\perp} \left(1 + \frac{2\pi i\rho'}{k_z} \int W(\vec{\eta}) d^3\vec{\eta} \right) e^{ik_z z} \tag{1.35}$$

(the wave is incident in the $+z$ direction, i.e., $k_z > 0$). If $z - z_0$ is much larger than the radius of action of the potential $W(\vec{\eta})$, but the observation point is in front of the plate, i.e., $z < z_0$, then

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}_\perp\vec{r}_\perp} \left[e^{ik_z z} + \frac{2\pi i\rho'}{k_z} \int W(\vec{\eta}) e^{i2k_z\eta_z} e^{i2k_z z_0} d^3\vec{\eta} e^{-ik_z z} \right] . \tag{1.36}$$

In fact, Eq. (1.35) describes a coherent wave which has passed through the plane, while Eq. (1.36) describes a wave mirror-reflected from the plane. From these two equations readily follows a well-known feature of refraction and mirror-reflection: the tangential components of a wave vector which are parallel to the surface of matter are equal in the incident, transmitted and reflected waves (see e.g. [Landau and Lifshitz (1984)]). It should also be noted that, according to Eq. (1.35), the amplitude of the transmitted wave is defined by the expression $\int W(\eta)d^3\eta$, which agrees with the first (Born) approximation for a forward scattering amplitude $f(0)$. The same result holds true beyond the scope of the perturbation theory.

At the same time, the amplitude of a reflected wave is defined by the expression:

$$\int W(\eta)e^{i2k_z\eta_z}d^3\eta = f(\vec{k}' - \vec{k}), \vec{k}'_\perp = \vec{k}_\perp, k'_z = -k_z\,,$$

i.e., by the amplitude of scattering at the angle equal to the doubled mirror reflection angle.

Now assume that we have a set of planes. Since, according to Eq. (1.35), as the wave passes through each plane, the amplitude changes by a factor of $1 + 2\pi i\rho' f(0)k_z^{-1}$, then after passing through m planes, the coherent wave will take the form:

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}_\perp\vec{r}_\perp}\left(1 + \frac{2\pi i\rho'}{k_z}f(0)\right)^m e^{ik_z z} \tag{1.37}$$

or

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}_\perp\vec{r}_\perp}e^{i\frac{2\pi\rho' m}{k_z}f(0)}e^{ik_z z}\,. \tag{1.38}$$

Since the number of planes is $m = z/a$, where z is the distance passed by the wave in matter; a is the distance between the planes, then

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}_\perp\vec{r}_\perp}e^{i\frac{2\pi\rho}{k_z}f(0)z}e^{ik_z z}\,, \tag{1.39}$$

where $\rho = \rho'/a$ is the volume density of matter.

Finally, Eq. (1.39) reads

$$\langle\psi(\vec{r})\rangle = e^{i\vec{k}_\perp\vec{r}_\perp}e^{ik_z nz}\,, \tag{1.40}$$

where

$$n = 1 + \frac{2\pi\rho}{k_z^2}f(0) \tag{1.41}$$

is the refractive index.

The expression obtained for n is valid when

$$\left|\frac{2\pi\rho}{k_z^2}f(0)\right| \ll 1\,.$$

If this inequality is violated, the expression for n appears to have the form:

$$n^2 = 1 + \frac{4\pi\rho}{k_z^2}f(0)\,. \tag{1.42}$$

If the scatters of different kind are present in matter, then

$$n^2 = 1 + \frac{4\pi}{k_z^2}\sum_i \rho_i f_i(0)\,, \tag{1.43}$$

where ρ_i is the number of i-type scatterers per 1 cm^3 of matter; $f_i(0)$ is the amplitude of coherent elastic zero-angle scattering by the i-type scatterer.

Thus, the microscopic summation of the waves scattered by atoms (nuclei) leads to the creation of a coherent wave moving in matter according to well-known classical laws of refraction. It is essential that in deriving Eq. (1.39) and Eq. (1.41) the requirement that a large number of scatterers be situated within the wavelength range was not used at all. Hence, the refractive index may also be introduced for studying radiation with a wavelength small in comparison with the distance between the scatterers.

Let us now turn to a mirror-reflected wave. According to Eq. (1.36), when the wave $e^{i\vec{k}_\perp\vec{r}_\perp}e^{ik_z z}$ is incident on the plane $z = z_0$, the amplitude of the mirror-reflected wave has the form:

$$A_1 = \frac{2\pi i\rho'}{k_z}f(\vec{k}' - \vec{k})e^{2ik_z z_0}\,.$$

If m planes fill the layer $[0, z]$, the amplitude of the wave scattered by these planes is equal to

$$A = \frac{2\pi i\rho'}{k_z}f(\vec{k}' - \vec{k})\sum_m e^{2ik_z z_m}\,. \tag{1.44}$$

Proceeding to continuous distribution of planes in the layer $[0, z]$, i.e., replacing summation by integration in Eq. (1.44), we finally obtain ($k_z z \gg 1$):

$$A = -\frac{\pi\rho}{k_z^2}f(\vec{k}' - \vec{k})\,. \tag{1.45}$$

It is seen that as a result of microscopic summation of the waves scattered in matter, there appeared the wave reflected in the direction determined by the laws of classical optics. The amplitude of this wave is determined by the amplitude of scattering at the angle equal to twice the angle of mirror reflection. Generally speaking, it cannot be obtained by conventional means – matching solutions at a sharp matter–vacuum boundary.

The conclusion about the existence of the phenomenon of mirror reflection of particles from the surface of matter can be drawn from some other considerations. For this purpose, let us pay attention to the fact that, due to the refraction effect, the kinetic energy $E_{\rm kin} = \hbar^2(k_\perp^2 + k_z^2 n^2)/(2m)$ of a coherent wave in a target is not equal to that $E_{\rm vac} = \hbar^2(k_\perp^2 + k_z^2)/(2m)$ of a wave in a vacuum.

As the state of the target does not change when an elastic coherent wave is produced, then, due to the energy conservation law, the aforesaid means that a particle in matter also possesses some effective potential energy

$$U = E_{\rm vac} - E_{\rm kin} = \frac{\hbar^2 k_z^2}{2m}(1 - n^2) = -\frac{2\pi\hbar^2}{m}\sum_i \rho_i f_i(0)\,. \tag{1.46}$$

Depending on the sign of $f(0)$, the energy U can be either larger or smaller than zero, i.e., the vacuum-matter boundary may be either a potential barrier ($U > 0$) or a well ($U < 0$). From the classical viewpoint, it is clear that if the particle velocity v_z satisfies the condition $mv_z^2/2 \leq U$, then at $U > 0$ the particle will undergo elastic reflection from a potential barrier. If the particle energy $E \gg U$ (typical of thermal neutrons), $mv_z^2/2 \approx mv^2\vartheta^2/2$, where ϑ is the glancing angle, and the phenomenon of the reflection from the barrier in the classical case occurs when ϑ is smaller than the critical angle

$$\vartheta_{\rm cr} = \sqrt{U/E} \tag{1.47}$$

As we have shown above, a mirror–reflected wave is necessarily created. The physical meaning of the angle $\vartheta_{\rm cr}$ is that at $\vartheta \leq \vartheta_{\rm cr}$, when $U > 0$, the coefficient of mirror reflection becomes equal to unity, i.e., total reflection occurs.

1.3 The Optical Theorem

Let a particle beam of intensity j_0 be incident normally to a plane surface of matter. Since $|\langle\psi(\vec{r})\rangle|^2$ is the probability density to find the particle at

point $\vec{r}$, the intensity of the particle beam at this point is

$$j = j_0 |\langle \psi(\vec{r}) \rangle|^2 = j_0 e^{-2k\mathtt{Im}nz} \,. \tag{1.48}$$

Let the matter have a form of a plane–parallel plate of thickness l and input surface area S. Let the plate be so thin as to satisfy the inequality $2k\mathtt{Im}nl \ll 1$. In this case

$$j = j_0(1 - 2k\mathtt{Im}nl) \,. \tag{1.49}$$

Multiplication of Eq. (1.49) by S gives the number of particles passing through the plate

$$N_{\rm pas} = N_0(1 - 2k\mathtt{Im}nl) = N_0 - N_0 2k\mathtt{Im}nl \,, \tag{1.50}$$

where N_0 is the number of particles incident on the plate.

The difference between N_0 and $N_{\rm pas}$ is apparently caused by the decrease in the number of particles in the initial beam occurring through scattering. Hence, the second term in Eq. (1.50) describes, in fact, the number of particles scattered by the plate.

There is another way to find this number. Remembering the definition in Eq. (1.1), we obtain that the total number of particles scattered by a particular center is equal to

$$N_{\rm 1scat} = \int j_{\rm scat} r^2 d\Omega = j_0 \int \frac{d\sigma}{d\Omega} d\Omega = j_0 \sigma \,, \tag{1.51}$$

where

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

is the total scattering cross section.

If the number of scattering centers in the plate is $N_{\rm c}$, then

$$N_{\rm scat} = N_{\rm 1scat} N_{\rm c} \,.$$

As $N_{\rm c} = \rho S l$,

$$N_{\rm scat} = j_0 \sigma \rho S l \,, \tag{1.52}$$

i.e.,

$$N_{\rm scat} = N_0 \sigma \rho l, \tag{1.53}$$

Comparing the second term in Eq. (1.50) with that in Eq. (1.53), we get

$$2k\mathtt{Im}n = \sigma\rho \,. \tag{1.54}$$

Recalling Eq. (1.41), we finally obtain

$$\mathtt{Im}f(0) = k\sigma/4\pi \,. \tag{1.55}$$

Equation (1.55) is called the "optical theorem". It appears to be suitable in the general case too, when inelastic processes and reactions are present. In this case the theorem indicates that the imaginary part of a coherent elastic zero-angle scattering amplitude is $k\sigma_{\rm tot}/4\pi$, where $\sigma_{\rm tot}$ is the total interaction cross section, including both elastic and inelastic processes.

In view of the law of absorption Eq. (1.48), which follows from the form of the wave function $\psi(\vec{r})$, it should be noted that in deriving it we actually used the fact that the wave function $\psi(\vec{r}) = e^{iknz}$ describes damping of the initial monochromatic beam with momentum $\vec{k}$ in the medium, and the perfect detector situated on the way of the primary beam only registers the particles with the initial momentum $\vec{k}$. Such a situation holds true with high accuracy, for example, for low-energy neutrons when the angular distribution of scattered particles is much greater than the angular resolution of a detector. However, in the high-energy range or for the Coulomb scattering of charged particles (protons, mesons), an appreciable fraction of particles is scattered at small angles and thus gets registered by the detector. In the limiting case when the angular distribution of scattered particles is much smaller than the angular resolution of the detector, all the particles will get into the detector and will be registered. As a consequence, no decrease in the number of particles in the initial beam will be registered, and the law Eq. (1.48) will be violated. In this case the transmission through matter of, for example, charged particles is described by the theory of multiple Coulomb scattering [Molière (1948); Bethe (1953); Bethe and Ashkin (1953); Ter-Mikaelian (1972)].

1.4 Scattering of Waves by a Set of Scatterers

Application of the optical theorem to the analysis of the process of scattering by a set of centers gives rise to a paradox ([Baryshevskii *et al.* (1965d)]). Indeed, let quite a slow particle with a wavelength much larger than both the size of the scatterers and the distance between these centers is elastically scattered by the two fixed identical centers. Let a denote the amplitude of particle scattering by each of these centers in the absence of the other.

As follows from Eq. (1.29), in this case the scattering amplitude does not depend on the scattering angle (i.e., on the direction of $\vec{k}'$). This result is true beyond the perturbation theory. That is why the amplitude of scattering at zero angle equals the amplitude of scattering at an arbitrary

angle. Then the optical theorem can be written as follows:

$$\mathtt{Im}a = k\sigma/4\pi = k|a|^2 \,. \tag{1.56}$$

It will be recalled that $d\sigma/d\Omega = |a|^2$. As a consequence, we get $\sigma = \int |a|^2 d\Omega = 4\pi|a|^2$.

Since the wavelength of the incident particle is assumed to be much larger than the distance between the scattering centers, it follows from Eq. (1.38) that in any direction the total scattering amplitude F is the sum of the amplitudes of scattering by particular centers, i.e., $F = 2a$. From this follows that the cross section is

$$\sigma' = 4\pi|2a|^2 = 4\sigma \,, \tag{1.57}$$

i.e., it is four times as large as the amplitude of scattering by a particular center.

On the other hand, if we try to find the cross section using the optical theorem, we obtain

$$\sigma' = \frac{4\pi}{k}\mathtt{Im}(2a) = 2\sigma \,, \tag{1.58}$$

i.e., the disagreement with the result obtained above is obvious.

The reason for this disagreement is that, strictly speaking, under given conditions we cannot consider the centers in question to have no influence on each other even if the amplitude of scattering by a particular center is $a \ll R$. Really, let one scatterer be situated at the origin of coordinates, while the other one, at point R. With due account of the fact that besides a primary plane wave, the wave scattered by the second center is also incident on the scatterer located at the origin of coordinates, the effective amplitude of scattering by the first center differs from the amplitude a of scattering by this center in the absence of the second scatterer and equals [1]

$$A_1 \approx a_1 + e^{i\vec{k}\vec{R}} a_2 \frac{e^{ikR}}{R} a_1 \,. \tag{1.59}$$

Here the phase factor $e^{i\vec{k}\vec{R}}$ appears due to the phase difference of the primary plane wave at the origin of coordinates and at point $\vec{R}$; $a_2 e^{ikR}/R$ is the wave which appears as a result of consecutive scattering of the primary plane wave by the second center and then by the first one.

[1] We used the fact that in isotropic scattering any wave is scattered with the same amplitude as a plane wave. Such a conclusion follows from the possibility to expand the waves in question in terms of plane waves with different momentum directions; the scattering amplitudes of these plane wave are independent of the angle.

With due account of the wave scattered by the first center, the amplitude of scattering by the second center has the form

$$A_2 \approx a_2 + e^{i\vec{k}\vec{R}} a_1 \frac{e^{ikR}}{R} a_2 \,. \tag{1.60}$$

Let us now calculate the imaginary part of the amplitude of forward scattering by the two centers $A(0) = A_1(0) + A_2(0)$:

$$\mathtt{Im}A(0) = \mathtt{Im}\,(A_1 + A_2) = \mathtt{Im}\left(a_1 + a_2 + \frac{2a_1 a_2}{R}\cos(\vec{k}\vec{R})e^{ikR}\right) . \tag{1.61}$$

If the conditions $kR \ll 1$ and $|a| < R$ are fulfilled, it is clear that $\mathtt{Re}a_{1,2} \gg \mathtt{Im}a_{1,2}$ and

$$\mathtt{Im}A(0) \approx \mathtt{Im}a_1 + \mathtt{Im}a_2 + 2k\mathtt{Re}a_1\mathtt{Re}a_2 \,. \tag{1.62}$$

If $a_1 = a_2 = a$, then $\mathtt{Re}a_1\mathtt{Re}a_2 \approx |a|^2$ and the imaginary part of the forward scattering amplitude is

$$\mathtt{Im}A(0) = 4\mathtt{Im}a \,,$$

i.e., the total cross section of scattering by the set of scatterers is really four times as large as that of scattering by a particular center.

Thus, taking account of rescattering eliminates the reviewed paradox as the additional terms associated with rescattering significantly change the imaginary part of the scattering amplitude, having practically no effect on its real part which is much larger than the imaginary one.

A more detailed analysis of scattering by two and more centers is given in [Brueckner (1953); Watson (1957); Baryshevsky (1995b)].

In this connection, similar phenomena in classical electrodynamics can be mentioned for comparison. Consider, for example, two neighboring oscillators excited by the light with a wavelength much larger than the distance between them. Then the equation of motion for the oscillators reads

$$\begin{aligned} \ddot{\vec{r}}_1 + \gamma(\dot{\vec{r}}_1 + \dot{\vec{r}}_2) + \omega_0^2\vec{r}_1 &= \frac{e}{m}\vec{\mathcal{E}}e^{-i\omega t} \,, \\ \ddot{\vec{r}}_2 + \gamma(\dot{\vec{r}}_1 + \dot{\vec{r}}_2) + \omega_0^2\vec{r}_2 &= \frac{e}{m}\vec{\mathcal{E}}e^{-i\omega t} \,. \end{aligned} \tag{1.63}$$

The term $\gamma\dot{\vec{r}}_2$ in the first equation describes the radiative reaction force of the second oscillator acting on the first one, while the term $\gamma\dot{\vec{r}}_1$ in the second equation describes the effect of the first oscillator on the second one.

Introduce new coordinates $\vec{R} = \vec{r}_1 + \vec{r}_2$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Then

$$\begin{aligned} \ddot{\vec{r}} + 2\gamma\dot{\vec{R}} + \omega_0^2\vec{R} &= \frac{2e}{m}\vec{\mathcal{E}}e^{-i\omega t} \,, \\ \ddot{\vec{r}} + \omega_0^2\vec{r} &= 0 \,. \end{aligned} \tag{1.64}$$

The second equation describes the undamped motion of the oscillators vibrating in antiphase. As would be expected, this motion results in neither radiation nor absorption of electromagnetic waves because in this case the dipole moment of the system does not change.

The effect of the external field on the oscillators is described by the first equation. Its stationary solution is:

$$\vec{R} = \left(\frac{2e}{m} \frac{1}{\omega_0^2 - \omega^2 - i2\omega\gamma} \right) \vec{\mathcal{E}} e^{-i\omega t} . \tag{1.65}$$

The term between the brackets is proportional to the amplitude of electromagnetic wave scattering by two oscillators. It is easy to notice that far from the resonance when the imaginary part can be neglected, the real part of the amplitude is proportional to $2e$, i.e., equals the sum of the amplitudes of scattering by independent oscillators. Far from the resonance, the imaginary part of the amplitude does not equal the sum of the amplitudes since the radiation from one of the oscillators has a significant influence on the decay of the other one, and vice versa.

Now let us consider scattering by two centers in more detail. Assume that scattering by each particular center is isotropic (s-scattering). The relation between the wavelength and the distance between the scatterers is supposed to be arbitrary. Let us first consider forward scattering. With due account of double scattering, the amplitude is

$$A_1^{(2)} \approx a_1 + a_1 a_2 \frac{e^{ikR}}{R} e^{i\vec{k}\vec{R}} + a_1 a_2 a_1 \frac{e^{2ikR}}{R^2} . \tag{1.66}$$

The term referring to the n-fold scattering has the form

$$a_1 a_2 a_1 a_2 \ldots a_1 a_2 \frac{e^{inkR}}{R^n} e^{i\vec{k}\vec{R}}$$

if n is an odd number. If n is an even number, then

$$a_1 a_2 \ldots a_1 a_2 a_1 \frac{e^{inkR}}{R^n}$$

(in both cases the number of amplitudes in the numerator equals $n + 1$).

For the second center

$$A_2^{(2)} \sim a_2 + a_2 a_1 \frac{e^{ikR}}{R} e^{-i\vec{k}\vec{R}} + a_2 a_1 a_2 \frac{e^{2ikR}}{R^2} . \tag{1.67}$$

The term referring to the n-fold scattering takes the form:

$$a_2 a_1 a_2 a_1 \ldots a_2 a_1 \frac{e^{inkR}}{R^n} e^{-i\vec{k}\vec{R}} \tag{1.68}$$

if n is an odd number. If n is an even number, it equals

$$a_2 a_1 \dots a_2 a_1 a_2 \frac{e^{inkR}}{R^n} \,. \tag{1.69}$$

It is quite obvious that the progressive approximations satisfy the relations

$$\left.\begin{aligned} A_1^{(n)} &= a_1 + a_1 A_2^{(n-1)} \tfrac{e^{ikR}}{R} e^{i\vec{k}\vec{R}} \,; \\ A_2^{(n)} &= a_2 + a_2 A_1^{(n-1)} \tfrac{e^{ikR}}{R} e^{-i\vec{k}\vec{R}} \,. \end{aligned}\right\} \tag{1.70}$$

At $n \to \infty$, $A_1^{(n)} \to A_1$ and $A_2^{(n)} \to A_2$, so

$$\left.\begin{aligned} A_1 &= a_1 + a_1 A_2 \tfrac{e^{ikR}}{R} e^{i\vec{k}\vec{R}} \,; \\ A_2 &= a_2 + a_2 A_1 \tfrac{e^{ikR}}{R} e^{-i\vec{k}\vec{R}} \,. \end{aligned}\right\} \tag{1.71}$$

Solving the system of equation, we get

$$\begin{aligned} A_1 &= \frac{a_1 + a_1 a_2 \dfrac{e^{ikR}}{R} e^{i\vec{k}\vec{R}}}{1 - a_1 a_2 \dfrac{e^{2ikR}}{R^2}} \,; \\ A_2 &= \frac{a_2 + a_1 a_2 \dfrac{e^{ikR}}{R} e^{i\vec{k}\vec{R}}}{1 - a_1 a_2 \dfrac{e^{2ikR}}{R^2}} \,. \end{aligned} \tag{1.72}$$

For scattering at an arbitrary angle, the effective amplitude A is associated with A_1 and A_2 by a simple relation

$$A = A_1 + A_2 e^{-i\vec{q}\vec{R}} \,, \tag{1.73}$$

where $\vec{q}$ is the transmitted momentum.

Equation (1.72) implies that each scatterer still scatters isotropically with the amplitudes defined by Eq. (1.72). In deriving Eq. (1.71) we supposed that the sequences $A_1^{(n)}$ and $A_2^{(n)}$ have limits. Thus, in fact, we used the conditions $a_{1,2} < R$. However, Eq. (1.71) actually holds true for any relation between a and R. Indeed, the sum wave incident on each of the scatterers is composed of a primary plane wave and a certain effective wave that appears as a result of the presence of the second center. Scattering of a plane wave is described by the terms a_1 and a_2 in Eq. (1.71). The physical meaning of the second terms in these equations becomes clear if remembering that in isotropic scattering any wave is scattered with the same amplitude as the plane one. Each of these terms describes the wave

produced by one of these centers and then scattered with the amplitude a by the other center. Since at each stage scattering is isotropic, the effective amplitudes A_1 and A_2 also remain isotropic.

It follows from the above that the first equation Eq. (1.71) could be derived from the following considerations: The contribution to the amplitude from the plane wave is a_1; one should add to a_1 the additional term describing scattering by the first center with the amplitude a_1 of the effective wave

$$A_2 \frac{e^{ikR}}{R} e^{i\vec{k}\vec{R}} ,$$

which has been produced by the second center. The second equation Eq. (1.71) is obtained analogously. It is obvious that when deriving the system of equations in this way, no restrictions on the relation between a and R are imposed.

The system of equations Eq. (1.71) is easy to generalize to the case of several scatterers [Baryshevskii *et al.* (1965c)]. In this case, the effective amplitude A of scattering by many fixed centers is

$$A = \sum_i A_i e^{-\vec{q}\vec{R}_i} , \tag{1.74}$$

and A_i satisfies the following system of equations

$$A_i = a_i + a_i \sum_{k \neq i} A_k \frac{e^{ikR_{ik}}}{R_{ik}} e^{i\vec{k}\vec{R}_{ki}} , \tag{1.75}$$

where $\vec{R}_{ki} = \vec{R}_k - \vec{R}_i$ and summation is made over all scatterers.

Let now along with elastic scattering, inelastic processes be possible. If $a \ll R$, we can confine ourselves with high accuracy to taking into account only one rescattering event. In this case the amplitudes A_1 and A_2 of elastic scattering by a set of two centers are still described by relations Eq. (1.71), but the difference is that now a stands for the amplitude including inelastic processes too:

$$\begin{aligned} \mathtt{Im}a &= \frac{k}{4\pi}(\sigma_{\rm el} + \sigma_{\rm inel}) , \\ \sigma_{\rm el} &= 4\pi|a|^2 . \end{aligned} \tag{1.76}$$

With more rescattering events taken into account, the expressions for the amplitudes A_1 and A_2 differ from those in Eq. (1.72) already in the order of $(a/R)^2$. The terms of the type as below appear in the amplitudes:

$$b_1 \frac{e^{ik'R}}{R} a_2 \frac{e^{ik'R}}{R} \bar{b}_1 , \tag{1.77}$$

where b_1 is the amplitude of the inelastic process at the first center (for example, inelastic scattering); $\bar{b}_1$ is the amplitude of the inverse process; k' is the wave number of the inelastically scattered wave.

These terms describe the processes due to which a wave is inelastically scattered by the first center, elastically rescattered by the second center, and when brought back to the first center returns this center into its initial state. Since after scattering the state of the system remains the same, such a process is to be considered as an elastic one. However, if the amplitude of the inelastic reaction $b \sim a \ll R$, the contribution of inelastic channels to the amplitude of elastic scattering by a set of centers through the above–described mechanism can also be neglected in higher orders. In this case Eq. (1.71), as well as Eq. (1.75), holds true for inelastic processes too, and a_1 stands for the elastic scattering amplitude, which obeys the relations Eq. (1.76). This statement immediately follows from the fact that the elastic scattering by a set in the presence of inelastic channels can be described, using a complex potential, which at $b \sim a \ll R$ is reduced to the sum of complex potentials of particular centers. Such substitution of a complex potential for a real one does not violate the validity of derivation of Eq. (1.75).

In the general case when the amplitude of the inelastic reaction is comparable with that of the elastic reaction, Eq. (1.75) becomes invalid.

Let us consider some particular cases. Let scattering be performed by two identical centers ($a_1 = a_2 = a$). Suppose that the cross sections of inelastic processes and elastic scattering have the same order of magnitude. In this case, at $|a| \ll R$ and $\lambda \gg R$, the inequality $\mathtt{Im}a \ll \mathtt{Re}a$ is fulfilled automatically, and we can confine ourselves to the first approximation Eq. (1.72) and Eq. (1.73). Then

$$\mathtt{Im}A \approx \mathtt{Im}\left(2a + \frac{2a^2}{R}e^{ikR}\right) \approx 2\mathtt{Im}a + 2k|a|^2 = \frac{k}{4\pi}(2\sigma^0_{\rm tot}) + 2k|a|^2\,, \quad (1.78)$$

where $\sigma^0_{\rm tot}$ is the total cross section of scattering by each center ($\sigma^0_{\rm tot} = \sigma_{\rm inel} + 4\pi|a|^2$). From this it follows

$$4\pi\mathtt{Im}A/k = 4\sigma^0_{\rm el} + 2\sigma^0_{\rm inel}\,, \quad (1.79)$$

i.e., the total cross section of scattering by two centers, as expected, can be presented as

$$\sigma_{\rm tot} = 2\sigma^0_{\rm inel} + 4\sigma^0_{\rm el}\,, \quad (1.80)$$

Now let us assume that $|a| \approx R$ and the system of equations Eq. (1.71) holds true. Then in the long–wave approximation

$$A \approx \frac{2a}{1 - ae^{ikR}/R}\,. \quad (1.81)$$

According to the optical theorem,

$$\sigma_{\rm tot} = \frac{4\pi}{k}\mathtt{Im}A \simeq \frac{2\sigma_{\rm inel} + 16\pi|a|^2}{1 + |a|^2/R^2 - 2\mathtt{Re}a/R}$$
$$= \frac{2(\sigma^0_{\rm inel} + 2\sigma^0_{\rm el})}{1 + \dfrac{\sigma^0_{\rm el}}{4\pi R^2} \pm \dfrac{2}{R}\sqrt{\dfrac{\sigma^0_{\rm el}}{4\pi} - \dfrac{(k\sigma^0_{\rm tot})^2}{16\pi^2}}}\,. \tag{1.82}$$

where σ^0 is the interaction cross section for a single center.

When $\sigma^0_{\rm el} \ll R^2$, we obtain Eq. (1.80). In the inverse limiting case when $\sigma^0_{\rm el} \gg R^2$, the elastic scattering cross section is [2]

$$\sigma_{\rm el} = 16\pi R^2\,. \tag{1.83}$$

If inelastic processes occur, from Eq. (1.82) follows that

$$\sigma_{\rm tot} = 16\pi R^2\left(1 + \frac{\sigma^0_{\rm inel}}{2\sigma^0_{\rm el}}\right)\,. \tag{1.84}$$

In particular, for scattering by absolutely black spheres ($\sigma_{\rm el} = \sigma_{\rm inel}$)

$$\sigma_{\rm tot} = 24\pi R^2\,. \tag{1.85}$$

Now let us give a more rigorous consideration of the process of creation of a coherent wave passing in matter. For this purpose, we shall study the problem of scattering by N centers in the general form. The corresponding Schrödinger equation is

$$\left(E_a - H(\xi_1 \dots \xi_N) + \frac{\hbar^2}{2m}\Delta_r\right)\psi(\vec{r}, \xi_1 \dots \xi_N)$$
$$= \sum_{i=1}^{N} V_i(\vec{r}, \xi_i)\psi(\vec{r}, \xi_1 \dots \xi_N)\,, \tag{1.86}$$

where $H(\xi_1 \dots \xi_N)$ is the Hamiltonian of the scatters; $\xi_1 \dots \xi_N$ is the set of coordinates describing the first and other scatterers (ξ also includes spin variables); $V_i(\vec{r}, \xi_i)$ is the energy of the interaction between the incident particle and the i-th scatterer; $\vec{r}$ is the coordinate of the incident particle.

If $G(\vec{r}, \xi_1 \dots \xi_N; \vec{r}\,', \xi'_1 \dots \xi'_N)$ is the Green function of the operator

$$E_a - H + \frac{\hbar^2}{2m}\Delta_r\,,$$

[2] Applying the optical theorem, one cannot drop the term e^{ikR} in the denominator of Eq. (1.72), though $kR \ll 1$. It is easy to see that if it is assumed that $e^{ikR} \simeq 1$, after using the optical theorem we obtain the value of the elastic scattering cross section which is half as large as the correct one.

then Eq. (1.86) can be written in the form

$$\psi_a(\vec{r},\xi_1\dots\xi_N)=\Phi_a(\vec{r},\xi_1\dots\xi_N)+\int\int G(\vec{r},\xi_1\dots\xi_N;\vec{r}',\xi_1'\dots\xi_N')\sum_{i=1}^{N}V_i(\vec{r}',\xi_i')\times\psi_a(\vec{r}',\xi_1'\dots\xi_N')d_3r'd_3\xi_1'\dots d_3\xi_N' \tag{1.87}$$

[Φ_a are the eigenfunctions of the operator $\left(-\frac{\hbar^2}{2m}\Delta_r+H\right)$].

Taking into account that [Goldberger and Watson (1984); Davydov (2010)]

$$\sum_i V_i(\vec{r},\xi_i)\psi_a(\vec{r},\xi_1\dots\xi_N)=T(\vec{r},\xi_1\dots\xi_N)\Phi_a(\vec{r},\xi_1\dots\xi_N)\,, \tag{1.88}$$

where T is the operator of scattering by N centers, the following equation can be derived for T:

$$T(\vec{r},\xi_1\dots\xi_N)\Phi_a(\vec{r},\xi_1\dots\xi_N)=\sum_{i=1}^{N}V_i(\vec{r},\xi_i)\Phi_a(\vec{r},\xi_1\dots\xi_N)+\sum_{i=1}^{N}V_i(\vec{r},\xi_i)\int\int G(\vec{r},\xi_1\dots\xi_N;\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)\times T(\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)\Phi_a(\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)d^3\rho d^3\eta_1\dots d_N^{\eta}\,. \tag{1.89}$$

Let us introduce the notation $T(\vec{r},\xi_1\dots\xi_N)\Phi_a(\vec{r},\xi_1\dots\xi_N)=T_a(\vec{r},\xi_1\dots\xi_N)$. Then, it is convenient to introduce the operators T^i, using the equalities:

$$T_a^{(i)}(\vec{r},\xi_1\dots\xi_N)=V_i(\vec{r},\xi_i)\Phi_a(\vec{r},\xi_1\dots\xi_N)+V_i(\vec{r},\xi_i)\int\int G(\vec{r},\xi_1\dots\xi_N;\,\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)\times T_a(\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)d^3\,\rho d^3\eta_1\dots d^3\eta_N\,, \tag{1.90}$$

i.e., $T=\sum\limits_i T^{(i)}$.

The system Eq. (1.89) can be represented as

$$T_a^{(i)}(\vec{r},\xi_1\dots\xi_N)=t_a^{(i)}(\vec{r},\xi_1\dots\xi_N)+t^{(i)}(\vec{r},\xi_1\dots\xi_N)\int\int G(\vec{r},\xi_1\dots\xi_N;\,\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)\times\sum_{l\neq i}T_a^{(l)}(\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)d^3\rho d^3\eta_1\dots d^3\eta_N\,, \tag{1.91}$$

where

$$t_a^{(i)}(\vec{r},\xi_1\dots\xi_N) = V_i(\vec{r},\xi_i)\Phi_a(\vec{r},\xi_1\dots\xi_N)$$
$$+V_i(\vec{r},\xi_i)\int\int G(\vec{r},\xi_1\dots\xi_N;\,\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)$$
$$\times t_a^{(i)}(\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)d^3\rho d^3\eta_1\dots d^3\eta_N\,.$$

As is known [Goldberger and Watson (1984); Davydov (2010)],

$$G(\vec{r},\xi_1\dots\xi_N;\,\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)$$
$$= -\frac{m}{2\pi\hbar^2}\sum_b \varphi_b(\xi_1\dots\xi_N)\varphi_b^*(\vec{\eta}_1\dots\vec{\eta}_N)\frac{e^{ik_b|\vec{r}-\vec{\rho}|}}{|\vec{r}-\vec{\rho}|}\,,$$

where $\varphi_b(\xi_1\dots\xi_N)$ are the eigenfunctions of the operator $H(\xi_1\dots\xi_N)$;

$$k_b^2 \equiv \frac{2m}{\hbar^2}\left(E_A + \frac{\hbar^2 k_a^2}{2m} - E_B\right) = \frac{2m}{\hbar^2}(E_a - E_B)\,;$$

E_A and E_B are the internal energies of the scattering system before and after the collision, respectively.

If the scatterers are independent of each other, the wave function $\varphi_b(\xi_1\dots\xi_N)$ is represented as the product of the wave functions of the scatterers:

$$\varphi_b(\xi_1\dots\xi_N) = \prod\varphi_{bi}(\xi_i)\,.$$

In this case the direct substitution can verify that $t^{(i)}(\vec{r},\xi_1\dots\xi_N) = t^{(i)}(\vec{r},\xi_i)$, where $t^{(i)}(\vec{r},\xi_i)$ is the operator of particle scattering by the i-th center in the absence of other centers.

Note now that the quantities $T_a^{(i)}$ can be written as follows:

$$T_a^{(i)} = t^{(i)}(\vec{r},\xi_1\dots\xi_N)F_a^{(i)}(\vec{r},\xi_1\dots\xi_N)\,, \tag{1.92}$$

where

$$F_a^{(i)}(\vec{r},\xi_1\dots\xi_N) = \Phi_a(\vec{r},\xi_1\dots\xi_N) + \int\int G(\vec{r},\xi_1\dots\xi_N;\,\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)$$
$$\times\sum_{l\neq i} T_a^{(l)}(\vec{\rho},\vec{\eta}_1\dots\vec{\eta}_N)d^3\rho d^3\eta_1\dots d^3\eta_N\,. \tag{1.93}$$

The first term in Eq. (1.93), being the function of $\vec{r}$, describes the initial wave falling upon the i-th scatterer. The second term can be interpreted as the contribution to the wave incident on the i-th center that is due to scattering by other centers. Indeed, if the interaction of the incident particle with all the centers excepting for the i-th center equaled zero, then the second term would also equal zero.

Let us make use of the definition Eq. (1.88) and rewrite equation Eq. (1.87) for the wave function ψ_a in the form:

$$\psi_a(\vec{r},\xi\dots\xi_N) = \Phi_a(\vec{r},\xi_1\dots\xi_N) + \int\int G(\vec{r},\xi_1\dots\xi_N;\vec{r}',\xi_1'\dots\xi_N')\sum_{i=1}^{N} t^{(i)}(\vec{r}',\xi_1'\dots\xi_N') \times F_a^{(i)}(\vec{r}',\xi_1'\dots\xi_N')d^3r'd^3\xi_1'\dots d^3\xi_N' . \quad (1.94)$$

From Eq. (1.94) follows that the probability amplitude $\psi_{ba}(\vec{r}) = \langle\varphi_b|\psi_a\rangle$ to find the particle at point $\vec{r}$ and the system in state b satisfies the system of equations

$$\psi_{ba}(\vec{r}) = e^{ik_a\vec{r}}\delta_{ba} - \frac{m}{2\pi\hbar^2}\int\int\frac{e^{ik_b|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\varphi_b^*(\xi_1'\dots\xi_N') \times\sum_t t^{(i)}(\vec{r}',\xi_1'\dots\xi_N')F_a^{(i)}(\vec{r}',\xi_1'\dots\xi_N')d^3r'd^3\xi_1'\dots d^3\xi_N' . \quad (1.95)$$

Transformation of Eq. (1.95) into a differential equation gives

$$(\Delta_r + k_b^2)\psi_{ba}(\vec{r}) - \frac{2m}{\hbar^2}\int\int\varphi_b^*(\xi_1'\dots\xi_N') \times\sum_{i=1}^{N} t^{(i)}(\vec{r},\xi_1\dots\xi_N)F_a^{(i)}(\vec{r},\xi_1\dots\xi_N)d^3\xi_1\dots d^3\xi_N = 0 \quad (1.96)$$

or

$$(\Delta_r + k_b^2)\psi_{ba}(\vec{r}) - \frac{2m}{\hbar^2}\sum_f\sum_{i=1}^{N} t_{bf}^{(i)}(\vec{r})F_{fa}^{(i)}(\vec{r}) = 0 , \quad (1.97)$$

where

$$t_{bf}^{(i)}(\vec{r}) = \int\varphi_b^*(\xi_1\dots\xi_N)t^{(i)}(\vec{r},\xi_1\dots\xi_N)\varphi_a(\xi_1\dots\xi_N)d^3\xi_1\dots d^3\xi_N ;$$

$$F_{fa}^{(i)}(\vec{r}) = \int\varphi_f^*(\xi_1\dots\xi_N)F_a^{(i)}(\vec{r},\xi_1\dots\xi_N)d^3\xi_1\dots d^3\xi_N .$$

Let us consider in more detail the equation describing the elastically scattered wave:

$$(\Delta_r + k_a^2)\psi_{aa}(\vec{r}) - \frac{2m}{\hbar^2}\sum_{i=1}^{N} t_{aa}^{(i)}(\vec{r})F_{aa}^{(i)}(\vec{r}) - \frac{2m}{\hbar^2}\sum_{f\neq a}\sum_{i=1}^{N} t_{af}^{(i)}(\vec{r})F_{fa}^{(i)}(\vec{r}) = 0 . \quad (1.98)$$

The amplitude $F_{fa}^{(i)}$ in the third term (unlike $F_{aa}^{(i)}$) appears only as a result of rescattering [see the general expression Eq. (1.93)]. For this reason, under the conditions when the elastic scattering amplitude f is of the same order of magnitude as the inelastic scattering amplitude and much smaller than the distance between the scatterers, the third term in the relation f/R for correlated scatterers and $(f/R)^2$ for independent scatterers is smaller than the second term, and can be discarded.

For simplicity, let us next study the phenomenon of refraction and reflection at the plane boundary of matter composed of scatterers independently vibrating about certain fixed equilibrium positions. Suppose also that all the oscillators are in the ground state [the wave function $\varphi_0(\vec{u}_i)$, where $\vec{u}_i$ is the displacement of the i-th center from the equilibrium position located at point $\vec{R}_i$]. Assume that $t^i(\vec{r}, \xi_1 \dots \xi_N) = t(\vec{r} - \xi_i)$. This occurs, for example, when the first Born approximation is applicable to the scattering operator and in the case of S–scattering when

$$t(\vec{r}, \xi_i) = -\frac{2\pi\hbar^2}{m} a\delta(\vec{r} - \xi_i)$$

where a is the scattering amplitude.

As a result, Eq. (1.98) can be presented in the form:

$$(\Delta_r + k_0^2)\psi_{00}(\vec{r}, \vec{R}_1 \dots \vec{R}_N) \tag{1.99}$$

$$-\frac{2m}{\hbar^2} \sum_{i=1}^{N} \int \varphi_0^*(\vec{u}) t(\vec{r} - \vec{R}_i - \vec{u}_i)\varphi_0(\vec{u}) d^3u F_0^{(i)}(\vec{r}, \vec{R}_1 \dots \vec{R}_N) = 0\,.$$

Now the functions ψ and F are the parametric functions of the coordinates R .

Let the oscillators be chaotically distributed with uniform density ρ in a half–space $R_z > 0$. The equation for a coherent wave $\langle\psi(\vec{r})\rangle$ is obtained by averaging Eq. (1.99) over the coordinates $\vec{R}_i$, which in our case is reduced to integration over $\vec{R}_i$ with the weight $1/V^N$ within the volume occupied by the medium

$$\langle\psi(\vec{r})\rangle = \frac{1}{V^N} \int\int \psi_{00}(\vec{r}, \vec{R}_1 \dots \vec{R}_N) d^3R_1 \dots d^3R_N\,, \tag{1.100}$$

i.e.,

$$(\Delta_r + k_0^2)\langle\psi(\vec{r})\rangle - \frac{2m}{\hbar^2} \sum_{i=1}^{N} \frac{1}{V} \int \varphi_0^*(\vec{u})$$

$$\times t(\vec{r} - \vec{R}_i - \vec{u})\varphi_0(\vec{u}) d^3u F_0^{(i)}(\vec{r}, \vec{R}_i) d^3R_i = 0\,, \tag{1.101}$$

$$F_0^{(i)}(\vec{r}, \vec{R}_i) = \frac{1}{V^{N-1}} \int\int F_0^{(i)}(\vec{r}, \vec{R}_1 \dots \vec{R}_{i-1}\,,$$

$$\vec{R}_i, \vec{R}_{i+1} \dots \vec{R}_N) d^3R_1 \dots d^3R_{i-1} d^3R_{i+1} \dots d^3R_N\,.$$

If there is a great number of scatterers that are randomly distributed, the addition or elimination of one of the scattering centers hardly affects the averaged sum wave passing in the medium [Lax (1951)]. This leads to the fact that the value of $F_0^i(\vec{r},\vec{R}_i)$ is, actually, independent of the coordinate $\vec{R}_i$ and coincides with $\langle\psi(\vec{r})\rangle$. Consequently, we can rewrite Eq. (1.101) as follows:

$$\left\{\Delta_r + k_0^2 - \frac{2m}{\hbar^2}\rho v(z)\right\}\langle\psi(\vec{r})\rangle = 0\,, \tag{1.102}$$

where

$$\rho v(z) = \rho\int_{R_z>0}\int \varphi_0^*(\vec{u})t(\vec{r}-\vec{R}-\vec{u})\varphi_0(\vec{u})d^3u d^3R$$

does not depend on the coordinates x and y due to the homogeneity of the medium along these coordinates. It has the meaning of the effective potential energy of interaction between the particle and the medium.

Let us consider Eq. (1.101) for positive values of the coordinate z, which are much larger than the oscillation amplitude a of the scatterers (the particle moves in the interior of the matter). At the given value of the coordinate z, the quantity

$$\int_{-\infty}^{+\infty}\int\left\{\int \varphi_0^*(\vec{u})t(\vec{r}-\vec{R}-\vec{u})\varphi_0(\vec{u})d^3u\right\}dR_x dR_y\,,$$

as the function of R_z is nonzero only in the layer with the thickness of the order of a. That is why for the stated values of z, the limits of integration with respect to $R_z[0,\infty]$ can be replaced by $[-\infty,\infty]$, i.e.,

$$v(z) = \int\int \varphi_0^*(\vec{u})t(\vec{r}-\vec{R}-\vec{u})\varphi_0(\vec{u})d^3u d^3R\,.$$

As the amplitude $f(\vec{k}'-\vec{k})$ of scattering by the center located at the origin of coordinates is related to the scattering operator t as

$$f(\vec{k}'-\vec{k}) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}'\vec{r}}\varphi_0^*(\vec{u})t(\vec{r}-\vec{u})e^{i\vec{k}\vec{r}}\varphi_0(\vec{u})d^3r d^3u\,, \tag{1.103}$$

then in the interior of the matter

$$v(z) = -\frac{2\pi\hbar^2}{m}f(0)\,.$$

In this case Eq. (1.102) describes the particle moving in the medium with the refractive index $n^2 = 1 + (4\pi\rho/k_z^2)f(0)$.

It is obvious that for $z < 0$ and $|z| \gg a$, Eq. (1.102) describes the particle moving in a vacuum.

Thus, the consideration of refraction and reflection at a matter boundary came to the investigation of refraction and reflection in the transition layer. Its degree of "diffusiveness" is determined by the vibration amplitude of the scatterers if the amplitude is much larger than the range of action of the potential and by the range of action of the potential if this range is greater than the vibration amplitude [Baryshevskii (1967)].

In the example discussed above we assumed that the equilibrium positions of the oscillators are uniformly distributed even near the surface. It is unlikely to be so for real substances. Even at an atomically clean boundary of a perfect crystal, the interaction between two or three boundary layers of atoms will be different from that in the interior of the crystal (it was shown in works on resonance absorption of γ-quanta in Mössbauer effect [Maradudin and Melngailis (1964); Bowles and Cranshaw (1965)] that the Debye–Waller factor of the surface nuclei is really different from that of the nuclei situated in the interior of the crystal). This causes the increase in the "diffusiveness" of the boundary up to the size of the order of one or two interatomic distances. Further growth of "diffusiveness" is due to roughness on the boundary.

In this case $\rho v(z)$ in Eq. (1.102) should be replaced by

$$\langle V(z)\rangle = \int\int \varphi_0^*(\vec{u})t(\vec{r}-\vec{u}-\vec{R})\varphi_0(\vec{u})\rho(\vec{R})d^3u d^3R\,,$$

where $\rho(\vec{R})$ is the density of the scatterers at the matter–vacuum boundary.

As is known, the coefficient of reflection from the layer with a diffuse edge depends significantly on the relation between the wavelength of the radiation and the size of the transition zone, as well as on the particle glancing angle. For example, for glancing angles and energies at which $k_z^2 \gg 4\pi\rho f(0)$ (k_z is the component of the particle wave vector along the z-axis), it is possible to apply the perturbation theory, which gives the following expression for the reflection coefficient:

$$R = \left| \frac{4\pi\rho}{|\vec{k}'-\vec{k}|^2} f(\vec{k}'-\vec{k}) \right|^2 , \tag{1.104}$$

where $\vec{k}'$ is the wave vector of the reflected wave. Using Eq. (1.103), Eq. (1.104) for the reflection coefficient can finally be presented as

$$R = R_0 \exp[-2W(k'-k)]\,, \tag{1.105}$$

where

$$R_0 = \left| \frac{4\pi\rho}{|\vec{k}'-\vec{k}|^2} f_0(\vec{k}'-\vec{k}) \right|^2$$

is the coefficient of reflection from the medium with rigidly fixed nuclei;

$$f_0(\vec{k}' - \vec{k}) = -\frac{m}{2\pi\hbar^2} \int e^{-i(\vec{k}'-\vec{k})\vec{r}} t(\vec{r}) d^3r$$

is the amplitude of scattering by a rigidly fixed center;

$$e^{-W(\vec{k}'-\vec{k})} = \int e^{-i(\vec{k}'-\vec{k})\xi} |\varphi_0(\xi)|^2 d^3\xi$$

is the form factor (the Debye–Waller factor).

As is seen, the angular dependence of the mirror reflection coefficient is determined by two factors: the anisotropy of scattering associated with the intrinsic characteristics of the scatterer [given by $f_0(\vec{k}' - \vec{k})$] and the anisotropy caused by the vibration of the scatterers.

Let us pay attention to the fact that the result obtained is significantly different from that obtained in deriving the coefficient of mirror reflection of the wave from the region characterized by an abrupt change in the refractive index (e.g. reflection from a sharp boundary between vacuum and matter). In the latter case, the coefficient of mirror reflection is defined by the equality like Fresnel's formula [Landau and Lifshitz (1984)] and, for example, for backward mirror reflection

$$R = \left| \frac{n-1}{n+1} \right| \approx \left| \frac{\pi\rho}{k^2} f(0) \right|^2 . \tag{1.106}$$

(In deriving Eq. (1.106) we used the conditions $n - 1 \ll 1$, $n + 1 \approx 2$.) On the other hand, it is clear from the physical viewpoint that a neutron (γ-quantum) wave mirror–reflected from a plane boundary of matter is a coherent superposition of waves scattered by nuclei (atoms) in the direction of reflection ϑ. Hence, the reflection coefficient R should depend on the amplitude of coherent elastic scattering by the center in the stated direction, i.e., on $f(\vartheta)$, but not on $f(0)$, which has actually been obtained [see formulas Eq. (1.104)–Eq. (1.105)] allowing for real "diffusiveness" of the boundary due to either the finite radius of action of the potential or the "diffusiveness" of the position of the nucleus (atom) as a result of vibrations. If the wavelength is much larger than the radius of the interaction potential (e.g. in the analysis of reflection of slow neutrons, whose wavelength is much larger than the radius of the nucleus), then the amplitude $f_0(\vec{k}' - \vec{k}) = f_0$ is a constant and the difference of the angular dependence of R from that in the case of rigidly fixed nuclei is determined by the form factor (the Debye–Waller factor). If the wavelength of the incident radiation is much larger than the nuclear vibration amplitude (much larger than the diffusiveness of the boundary), then even at backward reflection, the form

factor is equal to unity and relation Eq. (1.106) holds true. In particular, the applicability of Eq. (1.106) to light follows from the fact that for dipole scattering, the forward and backward scattering amplitudes are the same and the atomic recoil (diffusiveness of the boundary) can be neglected as the wavelength of light is large in comparison with a real diffusion of the atomically clean crystal surface.

Thus, for the wavelengths comparable with the dimensions of the transition zone (this occurs in the case of thermal neutrons and γ-quanta), the coefficient of reflection R depends on the type of this zone which is determined by nuclear dynamics.

It should be noted that the actual, principally unavoidable diffusiveness of the boundary can also manifest itself in the range of wavelengths of visible light in the phenomena, which arise due to the finite size of molecules (e.g. in optical activity (gyrotropy) of matter or other phenomena associated with spatial dispersion of matter). At any rate, the method of matching the solutions at a sharp (stepped) boundary between matter and vacuum applied to phenomenological analysis of gyrotropy (or other effects associated with spatial dispersion) should be analyzed from the microscopic viewpoint discussed above. The sensitivity of the reflection coefficient to the diffusiveness of the boundary enabled developing various research methods based on mirror reflection of neutrons, γ-quanta, and X-rays for studying the phenomena occurring in the transition zone between matter and vacuum [Felcher (1981); Deitrich and Wagner (1983); Vineyard (1982); Mazur and Mills (1982); Andreev (1985)].