XXIV International Seminar
Nonlinear Phenomena in Complex Systems
Minsk-Sosny, Belarus
May 16-19, 2017

Nonperturbative calculations in the framework of variational perturbation theory in QCD

Olga Solovtsova

ICAS , GSTU (Belarus)

Preface

The talk is devoted to the memory of I.L. Solovtsov

We present applications of the method based on the VPT * to perform calculations in QCD down to the lowest energy scale.

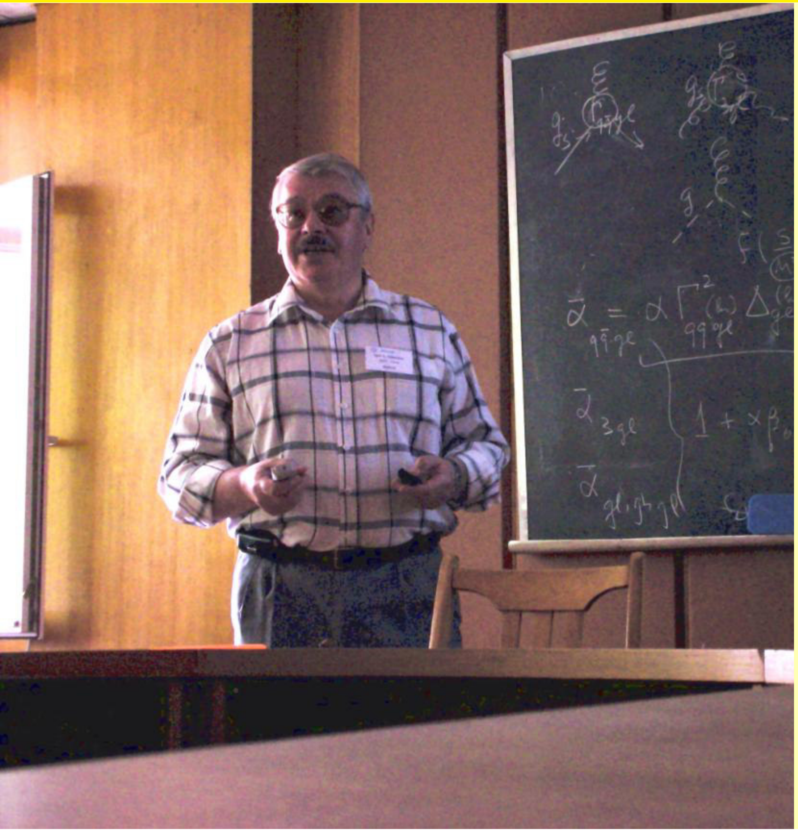
Last talk 10 years ago

* VPT (Variational Perturbative Theory)

✓ I.L. Solovtsov , «New expansions in QCD», Phys. Lett. B 327 (1994) 335

✓ «Nonperturbative expansions in QCD», Phys. Lett. B 340 (1994) 245

✓ A.N. Sissakian, I.L. Solovtsov, «Variational expansions in quantum chromodynamics», Phys. Part. Nucl. 30 (1999) 461



Sosny 2007

NONLINEAR PHENOMENA IN COMPLEX SYSTEMS

NPCS 2007

Nonperturbative Calculations in QCD, R-related Quantities and Operator Product Expansion

I. L. Solovtsov

ICAS, P. Sukhoy GSTU, Gomel, Belarus

Take care of Principles and
the Principles will take care of you.

D.V. Shirkov, I.L. Solovtsov

Method	Type of approximation	Properties		
		UV	IR	Anal
PT	Double set in powers of α_μ and $\ln Q^2/\mu^2$	$-$	$-$	$+$
PT + RG	Power series in invariant charge $\bar{\alpha}_s(Q^2)$	$+$	$-$	$-$
APT = PT + RG + analyticity	Nonpower expansions in $\mathcal{A}_k(Q^2)$ and $\mathfrak{A}_k(s)$			

Shirkov, Solovtsov 2007

From the theoretical point of view, the remarkable properties of the APT approach create a basis for the its preferable application.

Outline

- Overview of theoretical framework (low Q^2 scales):
  - timelike and spacelike characteristics of inclusive processes
  - R-related quantities
  - criterion of equivalence description
- Applications of VPTto hadrons physics
- Residual condensates
- DIS: VPT & Q^2– evolution SF:
  - QCD analysis of xF_3 -data
- Summary

Overview of theoretical framework:
timelike and spacelike characteristics

The main object in description of hadronic part of many physical processes is $\Pi(q^2)$

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx} < 0|T V_\mu(x) V_\nu(0)^+|0>$$
$$= (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(q^2)\ , \quad V_{ij}^\mu = \bar{\psi}_j \gamma^\mu \psi_i$$

It is useful to introduce an Eucledean characteristic, the so-called the Adler function

$$D(Q^2) = -Q^2 \frac{d\Pi(-Q^2)}{dQ^2}\ , \quad Q^2 = -q^2 > 0$$

[in Euclidean (spacelike) region]

The integral representation for the *D*-function is given in terms of the discontinuity of the correlator across the cut

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} R(s) \Rightarrow \quad R(s) = \frac{1}{2\pi i}[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)]$$

[in Minkowskian (timelike) region]

To parameterize R(s) in terms of QCD parameters a procedure of analytic continuation from Euclidean (s-channel) to Minkowskian (t-channel) region is required. The perturbative approximation, in which the running coupling with unphysical singularities is used, breaks the connection between space and timelike quantities. The VPT leads to a self-consistent definition of analytic continuation [H. Jones, I. Solovtsov, Phys. Lett. 349 (1995) 519].

R-quantities

We analyze various physical quantities and functions generated by *R(s)* based on the nonperturbative VPT-method. **A common feature of all these quantities and functions is that they are defined through the function *R(s)* integrated with some other function.**

- $$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right)R(s)$$

The ratio of hadronic to leptonic tau-decay widths in the vector channel

- $$a_l^{had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} K_l(s) R(s)$$

$l = \mu,\ e,\ \tau$

$K(s)$ - known QED kernel

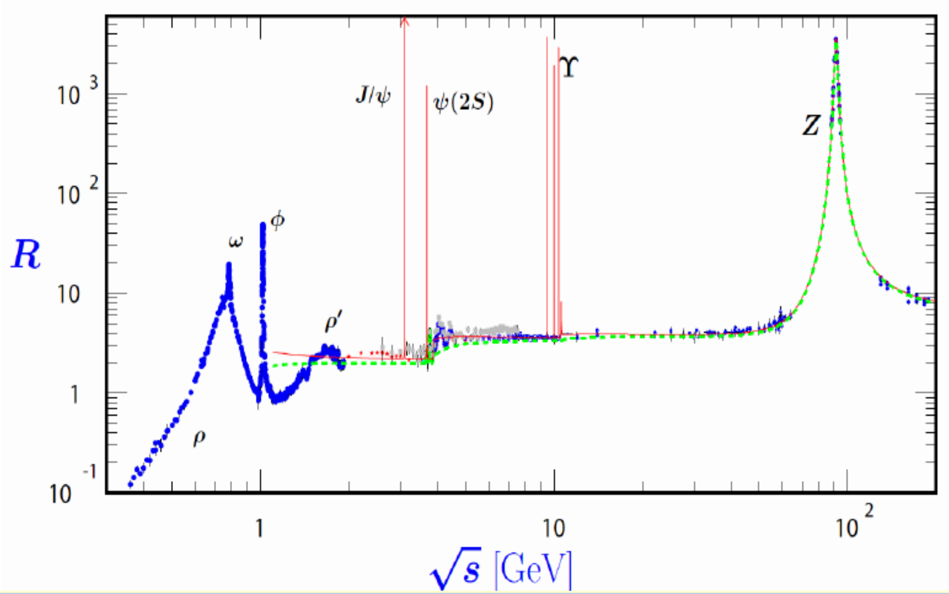
The hadronic contribution to the anomalous magnetic moment of the leptons
(in the leading order in the electromagnetic coupling constant)

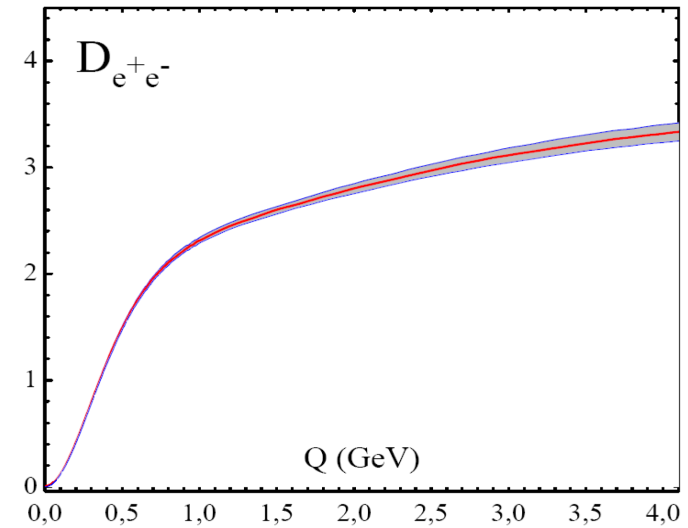
- $$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = -\frac{\alpha(0)}{3\pi} M_Z^2\, P\int_0^\infty \frac{ds}{s}\frac{R(s)}{s-M_Z^2}$$

The contribution to the running of the fine structure constant

All these quantities include an infrared region as a part of the interval of integration and, therefore, they cannot be directly calculated within perturbative QCD.

Clearly that it is fruitful to connect measured quantities with `simplest' theoretical objects. The Adler D-function (Euclidean quantity) turned out to be a smooth function without traces of the resonance structure of the R- ratio is a convenient object for comparing theoretical results with experimental data. The D-function have been considered in many works.





From F. Jegerlehner

S. Eidelman, F. Jegerlehner, A.I. Kataev, O. Veretin, Testing non-perturbative strong interaction effects via the Adler function, Phys. Lett. B (1999).

K. A. Milton, I. L. Solovtsov, O. P. S. Adler function for light quarks in analytic perturbation theory, Phys. Rev. D (2001).

A.E. Dorokhov, Adler function and hadronic contribution to the muon g-2 in a nonlocal chiral quark model, Phys. Rev. D (2004).

A.V. Nesterenko, On the low-energy behavior of the Adler function, Phys. Rev. D (2008).

F. Jegerlehner, The running fine structure constant via the Adler function, hep-ph:0807.4206 (2008).

B.A. Magradze, Testing the Concept of Quark-Hadron Duality with the ALEPH τ Decay Data, Few Body Syst. (2010).

T.Goecke, C.S.Fischer, R. Williams, Leading-order calculation of hadronic contributions to the muon g-2 using the Dyson-Schwinger approach, Phys. Lett. B (2011).

P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, J. Rittinger, Adler function, sum rules and Crewther relation of order $O(\alpha_s^4)$: the Singlet Case. Phys. Lett. B (2012).

VPT in QCD

- Within VPT method a quantity under consideration is represented by a power series with a new small expansion parameter a related to the original coupling constant $\lambda = \alpha_s/(4\pi)$ by the following equation

$$a^2 = C\lambda(1-a)^3$$

with a positive parameter C. This parameter is a variational parameter that controls properties of a variational or `floating' series. Clearly, the initial quantity does not depend on this parameters, however, any finite approximation does due to truncation of the series. A value of the parameter C can be found by using different ways.. For example, C can be defined from requirement of the Kallen-Lehmann analyticity of the $a(Q^2)$. Different ways lead to similar results.
Dealing with the accuracy $O(a^3)$ and taking the number of active quarks f=3 one finds $C \approx 4.0.$

The renormalization group beta-function of the expansion parameter is

$$\beta_a(a) = \mu^2 \frac{\partial a}{\partial \mu^2} = \frac{2\beta_0}{C} \frac{1}{f'(a)}, \qquad \beta_0 = 11 - 2f/3$$

The beta-function has a zero at $a=1$ that demonstrates the existence of the infrared fixed point of the expansion parameter and its freezing-like behavior in the infrared region.
The Q^2 **–evolution** for $a(Q^2)$:

$$\boxed{\ln \frac{Q^2}{Q_0^2} = \frac{C}{2\beta_0} \left[f(a) - f(a_0) \right]}$$

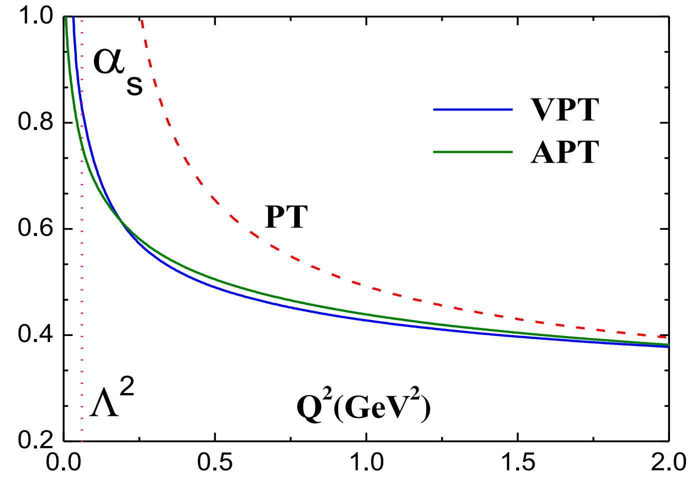
The parameter $a_0 = a(Q_0^2)$ is associated with some normalization point.

At the level $O(a^3)$, the function f(a) has the form

$$f^{(3)}(a) = \frac{2}{a^2} - \frac{6}{a} - 48\ln a - \frac{18}{11}\frac{1}{1-a} + \frac{624}{121}\ln(1-a) + \frac{5184}{121}\ln\left(1 + \frac{9}{2}a\right).$$

Remarks to the VPT approach

✓ The requirement of the Kallen-Lehmann representation (Q^2 –analyticity) for running coupling was used within the analytic perturbation theory (APT) [D.V. Shirkov, I.L.Solovtsov, Phys. Rev. Lett. 79 (1997)]
The APT leads to o very close to VPT results.
- VPT/APT free from unphysical singularities.
The QCD contribution calculated within the VPT/APT:
- better convergence properties and stability with respect to higher-loop corrections,
- practically invariant with respect to the choice of RS –prescription;



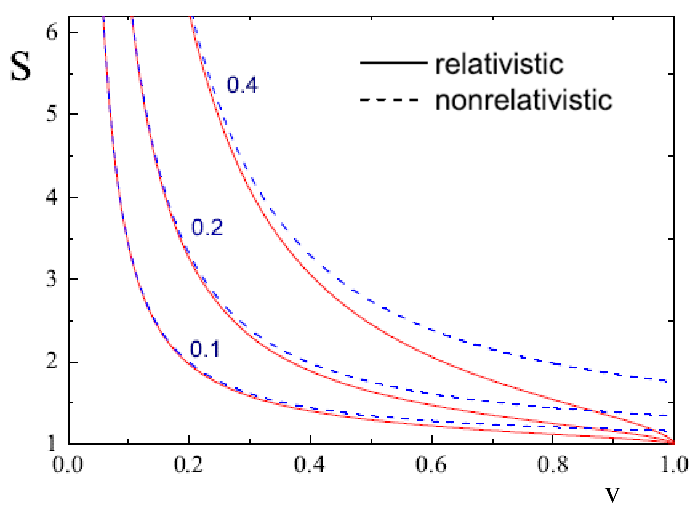
✓ The formulated model also incorporates a summation of threshold singularities and takes into account nonperturbative character of quark masses.

S-factor [Gamov-Sommerfeld-Sakhorov]

$$S_{nr}(\mathrm{v}) = \frac{X_{nr}(\mathrm{v})}{1-\exp[-X_{nr}(\mathrm{v})]}, \quad X_{nr}(\mathrm{v}_{nr}) = \frac{\pi\alpha}{\mathrm{v}_{nr}}$$

$$T(\mathrm{v}) = \mathrm{v}\frac{3-\mathrm{v}^2}{2}, \mathrm{v} = \sqrt{1-\frac{4m^2}{s}}, \quad X(\mathrm{v}) = \frac{\pi\alpha}{\mathrm{v}}\sqrt{1-\mathrm{v}^2}$$

$$R_0(s) = T(\mathrm{v})\, S(\mathrm{v}), \quad S_{nr}(\mathrm{v}_{nr}) \Rightarrow S(\mathrm{v})$$



Numerical results

Hadronic contribution
to the anomalous magnetic moment:

✓ ***muon*** $a_\mu^{had} = (694.9 \pm 3.7)\times 10^{-10}$ [one of set expt. result 2012]

$(702 \pm 16)\times 10^{-10}$ [VPT]

✓ ***electron*** $a_e^{had} = (1.678 \pm 0.014)\times 10^{-12}$ [Nomura, Teubner 2013]

$(1.64 \pm 0.07)\times 10^{-12}$ [VPT]

✓ ***tau lepton*** $a_\tau^{had} = (3.38 \pm 0.04)\times 10^{-6}$ [Passera'07, Nomura'2012]

$(3.28 \pm 0.05)\times 10^{-6}$ [VPT]

✓ ***fine structure*** $\Delta\alpha_{had}^{(5)}(M_Z^2) = (276.26 \pm 1.38)\times 10^{-4}$ [Hagivara et al. 2011]

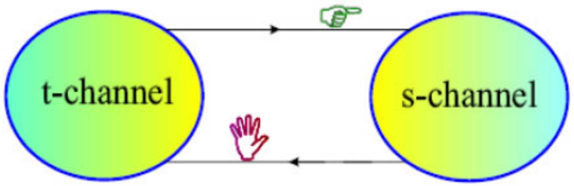
$(279.9 \pm 4.0)\times 10^{-4}$ [VPT]

Good agreement for all considered quantities has been obtained.
The question : Wy?

Criterion of equivalence

$$Q_M = \int_0^\infty \frac{ds}{s} M(s) R(s)$$

$$Q_E = \int_0^\infty \frac{dt}{t} E(t) D(t)$$



Minkowskian $\Longleftrightarrow$ Euclidean

$$Q_M = \int_0^\infty \frac{dt}{t} E(t) D(t) \equiv Q_E$$

(R-D self-duality)

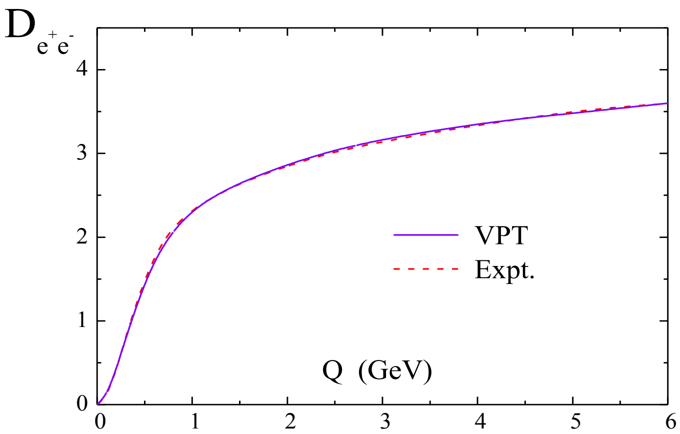
When expressions for quantity Q in terms of $R(s)$ and $D(Q^2)$ are equivalent?
The answer on the question about a simultaneous good agreement of various QCD observables is:
An approach used is support required analytic properties and gives good description of the D-function down to low energy scale.
If one uses a method that does not maintain the required analytic properties of functions then these expressions are not equivalent.

R-D self-duality presentations

$$a_\mu^{had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} M(s) R(s) \qquad a_\mu^{had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dt}{t} E(t) D(t)$$

$$E(t) = -\frac{1}{2\pi i}\int_{t-i\varepsilon}^{t+i\varepsilon} \frac{dz}{z} M(-z) \Rightarrow E(t) = \frac{1}{2}\left[\frac{\sqrt{1+4m^2/t}-1}{\sqrt{1+4m^2/t}+1}\right]^2,$$

$$M(s) == \int_0^1 dx \frac{x^2}{x^2 + (1-x)s/m_\mu^2} \qquad \eta = \frac{1-v}{1+v}, \qquad v = \sqrt{1-\frac{4m^2}{s}}.$$

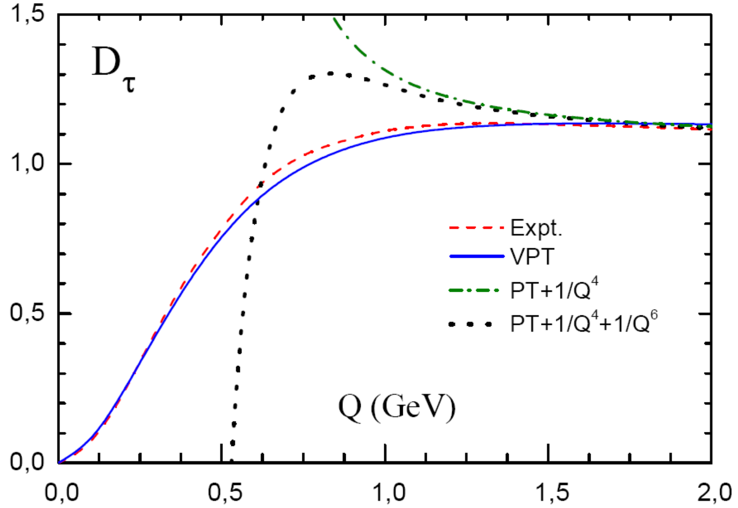


The theoretical approach (VPT) which we used to describe the experimental curve for the D-function works well (solid line) for the whole interval, including the infrared region.

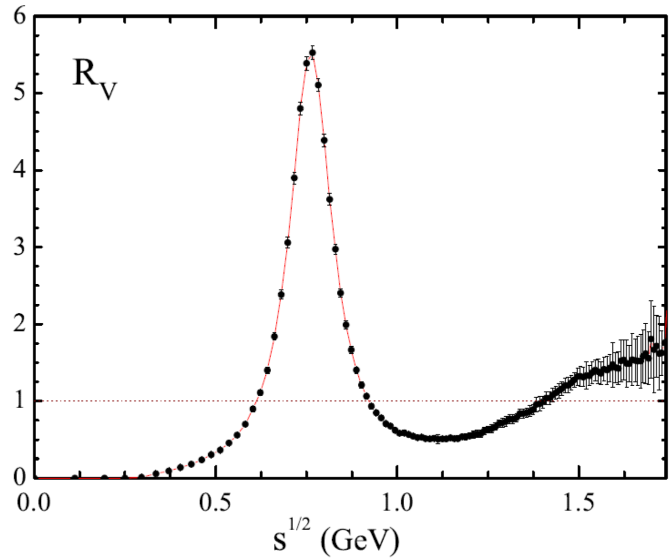
Figure from the paper Sissakian-Solovtsov, OS, A Nonperturbative a -Expansion Technique and the Adler D–function, JETP Letts. (2001).

Light D-function

The `light' Adler function (n_f=3) constructed from ALEPH tau-decay data



This figure is taken from the paper Sissakian-Solovtsov-OS, JETP Letts. (2001)



ALETH data

The experimental *D*-function (dashed curve) turned out to be a smooth and monotone function without traces of the resonance structure.

Note that any finite order of the operator product expansion (OPE) fails to describe the infrared tail of the *D*-function (dotted curve) .

OPE & condensates

The vacuum condensates play an important role in the QCD sum rule method [Shifman, Vainshtein, Zakharov (1979)]

Formally, the operator product expansion of the correlator $\Pi(q^2)$ of the vector quark current can be written as a sum of perturbative and nonperturbative parts

$$\Pi_{\rm OPE}(-Q^2) = \Pi_{\rm PT}(-Q^2) + \frac{\langle \mathcal{O}_2 \rangle}{Q^2} + \frac{\langle \mathcal{O}_4 \rangle}{Q^4} + \frac{\langle \mathcal{O}_6 \rangle}{Q^6} + \cdots$$

$\mathcal{O}_{2n}$ are the local operators constructed from quark and gluon fields.

Within the standard QCD sum rules approach $\langle \mathcal{O}_2 \rangle = 0$ since there is no gauge invariant operator of the dimension two.

The values of condensates, which are usually employed in phenomenological applications, are

$$\langle \mathcal{O}_4 \rangle_{\rm phen} \simeq 0.04\,{\rm GeV}^4, \quad \langle \mathcal{O}_6 \rangle_{\rm phen} \simeq -0.06\,{\rm GeV}^6.$$

Residual condensates

The difference between the experimental values and theoretical prediction of the correlator $\Pi(q^2)$ can be represented in the following form

$$\Delta\Pi(Q^2) = \Pi_{\rm expt}(Q^2) - \Pi_{\rm theor}(Q^2) \quad \Leftarrow \textit{ measure of knowledge/ignorance}$$

Dispersion relation

$$\Delta\Pi(Q^2) = \int_0^\infty ds\, \frac{\Delta R(s)}{s + Q^2},$$

$$\Delta R(s) = R_{\rm expt}(s) - R_{\rm theor}(s)$$

If one is able to derive the function $R_{\rm theor}$ exactly, in this case $\Delta R(s) = 0$.

One may expect that the method, which adequately describes the strong interaction processes in the infrared domain, inherently incorporates the most important condensates.

Assuming that the perturbative part of the correlator is reproduced good enough, one can represent the function $\Delta\Pi(-Q^2)$ as the sum of the residual nonperturbative contributions

$$\Delta\Pi(-Q^2) = \Pi_{\mathrm{expt}}(-Q^2) - \Pi_{\mathrm{theor}}(-Q^2) = \frac{O_2}{Q^2} + \frac{O_4}{Q^4} + \frac{O_6}{Q^6} + \cdots$$

Obviously, in this expression the "residual condensates" O_{2k} differ from above $\langle \mathcal{O}_{2k} \rangle$.

Similarly to the sum rule method, we apply the Borel transform, that eventually results in the basic sum rule

$$O_2 + \frac{O_4}{M^2} + \frac{O_6}{2M^4} = \mu_1(M^2),$$

where the Borel moments are

$$\mu_n(M^2) = \int_0^{s_0} ds\, s^{n-1} \exp(-s/M^2)\, \Delta R(s).$$

System of sum rules (can be derived by differentiating) gives the solution for the lowest dimension condensates O_2, O_4, and O_6:

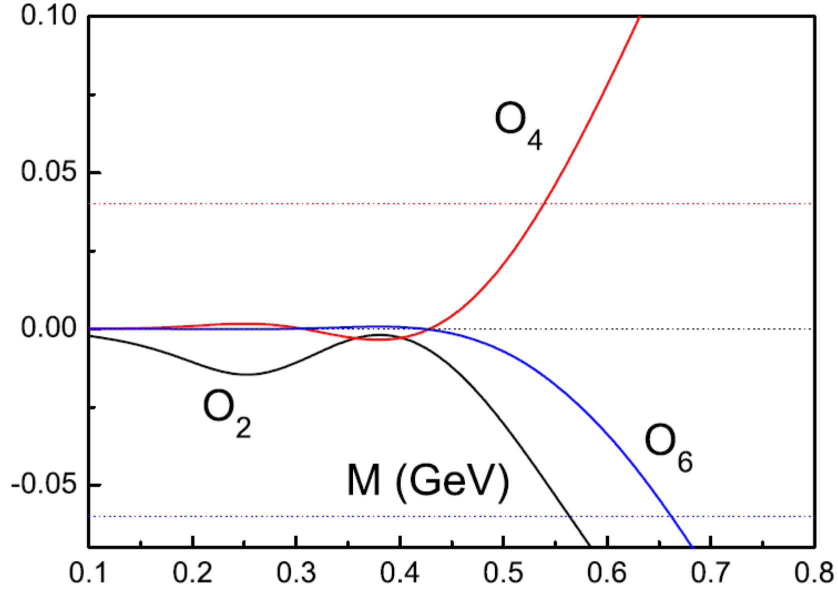
Estimations

$$O_2 = -2\,\mu_1(M^2) - \frac{\mu_2(M^2)}{2M^2} + 3\,\tilde{\mu}(M^2),$$

$$O_4 = 6\,M^2\mu_1(M^2) + 2\,\mu_2(M^2) - 6\,M^2\tilde{\mu}(M^2),$$

$$O_6 = -6\,M^4\mu_1(M^2) - 3\,M^2\mu_2(M^2) + 6\,M^4\tilde{\mu}(M^2)$$

$$\tilde{\mu}(M^2) = \int\limits_0^{s_0} ds\,\frac{1-\exp(-s/M^2)}{s/M^2}\,\Delta R(s)$$



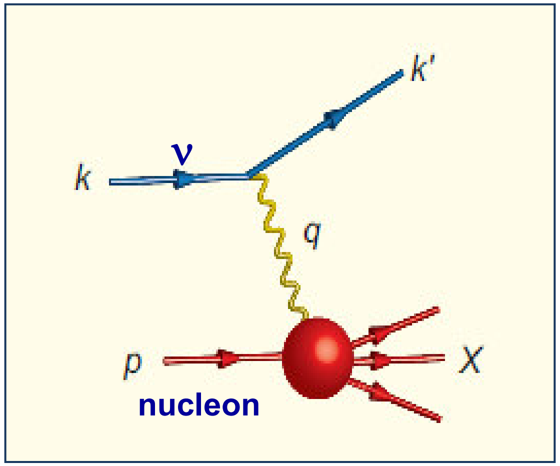
Horizontal lines correspond to the phenomenological values

$$\langle \mathcal{O}_4 \rangle_{\rm phen} \simeq 0.04\,{\rm GeV}^4, \quad \langle \mathcal{O}_6 \rangle_{\rm phen} \simeq -0.06\,{\rm GeV}^6.$$

The optimal values of O_2 are compatible with zero.

The model developed essentially accumulates the basic condensates of the lowest dimension.

Last application: QCD analysis of the xF_3



Deep inelastic scattering

The set of the various experimental data on the xF_3 structure function

- GARGAMELLE
- SCAT
- BEBC-WA59
- CDHS
- NuTeV
- CHORUS

289

The kinematic region of combined set of data is
$0.015 < x < 0.8$ and $0.5 \text{ GeV}^2 < Q^2 < 196 \text{ GeV}^2$

How the VPT approach works in comparison with the ordinary perturbative QCD?

Theoretical framework: Q^2– evolution SF

In the description of Q^2 -evolution of the structure function moments the generalized powers (anomalous dimensions) for the running coupling appear. Q^2 -evolution of the structure function moments in the nonsinglet case reads

$$M_N(Q^2) = \left[\frac{\alpha_S(Q^2)}{\alpha_S(Q_0^2)}\right]^{\nu} M_N(Q_0^2)\ , \qquad \nu(N) \equiv \gamma_{NC}^{0}(N)/2\beta_0 = -q^2 > 0$$

$$M_N(Q^2) = \int_0^1 x^{N-1} F_3(x, Q^2)\,dx$$

nonsinglet one-loop anomalous dimensions

$$M_N(Q_0^2) = \int_0^1 x^{N-2} A(Q_0^2) x^{\alpha} (1-x)^{\beta} (1+\gamma x)\,dx$$

In the framework of the VPT this expression transforms as follows:

$$\boxed{\mathcal{M}_N^{VPT}\left(Q^2\right) = \left[\frac{\lambda\left(Q^2\right)}{\lambda\left(Q_0^2\right)}\right]^{\nu} \mathcal{M}_N^{VPT}\left(Q_0^2\right)}$$

$$\lambda = \frac{1}{C}\,\frac{a^2}{(1-a)^3}, \qquad 0 \le a < 1$$

Results of fits

$$xF_3(x,Q_0^2) = Ax^{\alpha}(1-x)^{\beta}(1+\gamma x)$$

PT A, α, β, γ, Λ - free parameters

VPT A, α, β, γ, $a(Q_0^2)$ - free parameters

$$Q_0^2 = 3\ \mathrm{GeV}^2, \quad N_{max} = 11$$

	PT	VPT
A	**3.75 +- 0.58**	**3.83 +- 0. 53**
α	**0.647+- 0.043**	**0.648+- 0.040**
β	**3.62 +- 0.052**	**3.56+- 0.044**
γ	**2.75 +- 0.68**	**2.49 +- 0.60**

PT		VPT	
Λ (MeV)	χ^2/D.o.f	$a(Q_0^2)$	χ^2/D.o.f
363 +- 33	**519/289**	**0.279**	**515/289**

The shape of the structure function

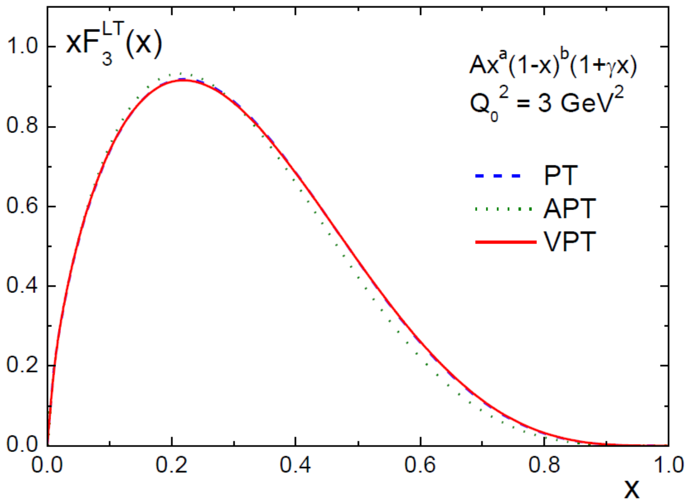
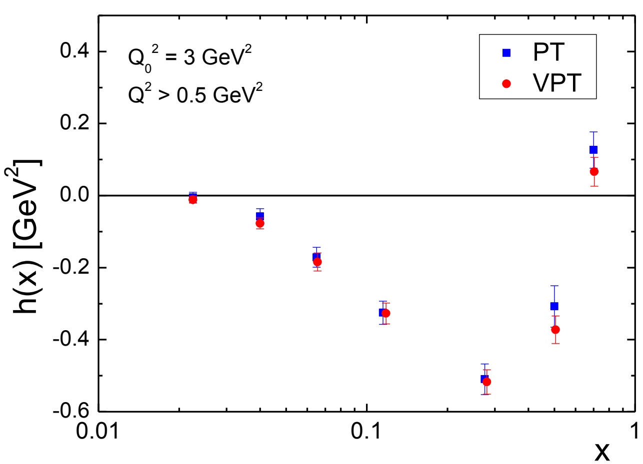
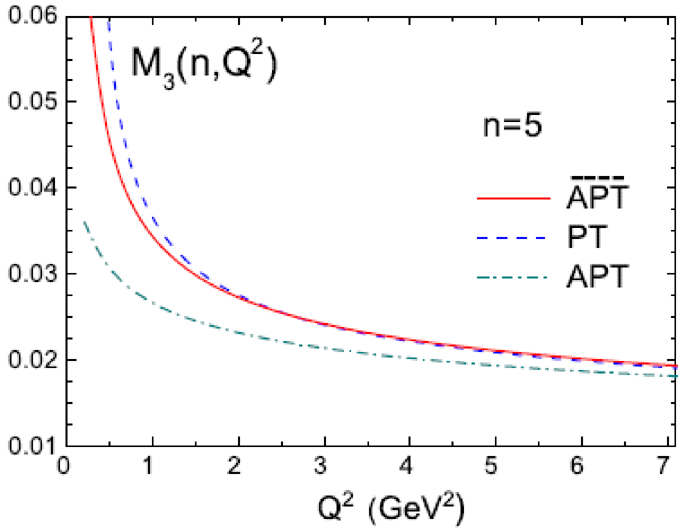


TABLE I: The results for the QCD leading order fit of a combined set on the xF_3 data in the PT and in the $\overline{\text{APT}}$ approaches at $Q_0^2 = 5$ GeV^2, $Q^2 > 0.5$ GeV^2.

	PT	$\overline{\text{APT}}$
A	3.40 ± 0.59	3.33 ± 0.56
α	0.60 ± 0.49	0.59 ± 0.47
β	3.78 ± 0.74	3.75 ± 0.77
γ	2.91 ± 0.80	2.90 ± 0.78
Λ [MeV]	351 ± 48	419 ± 67
$\chi^2_{d.f.}$	$514.5/292$	$511.1/292$



The shape of the higher twist



$$\Delta\left(xF_3^{PT}(x,Q^2)\right) = xF_3^{PT}(x,Q^2) - xF_3^{\overline{\mathrm{APT}}}(x,Q^2) \quad (12)$$

$$\Delta\left(xF_3^{APT}(x,Q^2)\right) = xF_3^{APT}(x,Q^2) - xF_3^{\overline{\mathrm{APT}}}(x,Q^2)\,. \quad (13)$$

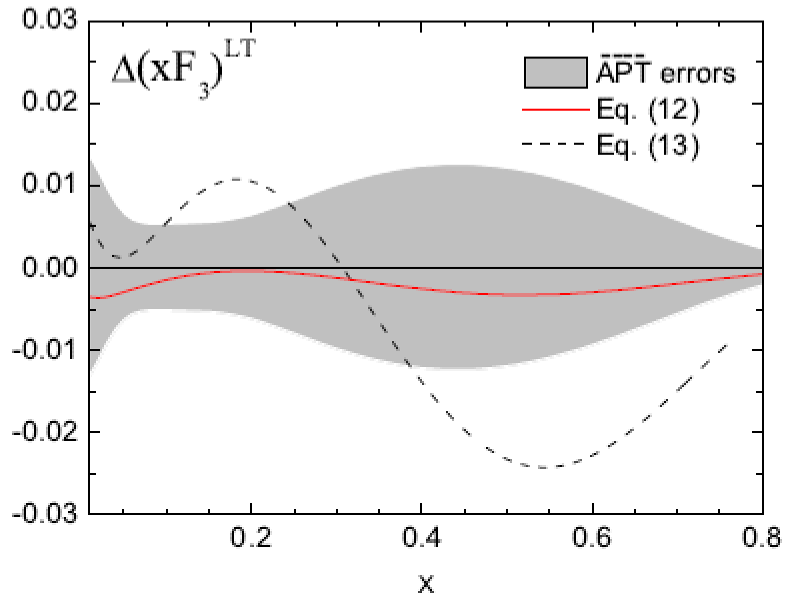


FIG. 1: The difference $\Delta(xF_3)$ for the PT (solid line), and for the APT(dash line) in comparison with uncertainties for $xF_3^{\overline{\mathrm{APT}}}$ (shaded area) at $Q_0^2 = 5$ GeV^2.

Summary

We have analyzed various physical quantities and functions generated by *R(s)* based on the nonperturbative VPT-method (Adler functions, hadronic contributions to anomalous magnetic moments of leptons and so on).
It was shown that the method allows us to describe these quantities rather well down to the lowest energy scale. The conventional method cannot describe the low energy region because both the logarithmic and power operator product expansions diverge at small momenta.
The Borel type sum rules which allow us to determine the residual condensates have been constructed. It was shown that within the method suggested the optimal values of these lower dimension condensates are close to zero. Therefore, the model includes at least important from phenomenological point of view condensates of lowest dimensions.
It was presented the result of the leading order QCD analyses of the structure function xF_3 data by using Q^2–evolution within the VPT method, which in contract to standard PT does not lead to any unphysical singularities.

Thanks for your attention !

May 2007

Жизнь сера?

Ни за что не поверю.

И любви нет?

Нашли дурака.

Нету счастья?

Нисколько не верю.

Жизнь прекрасна!

Но так коротка. (И. Л. Соловцов)

The End

I.L. Solovtsov

Thanks all for the attention again !